1. Consider the following subsets of $\mathbb{R}^3$. Determine which are open and which are closed in $\mathbb{R}^3$. [Fully justify your answers.]

   (i) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 5\}$,
   (ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 16 \text{ or } z > 3\}$,
   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 16 \text{ and } z \leq 3\}$,
   (iv) $\{(x, y, z) \in \mathbb{R}^3 : y > 0 \text{ and } x^2 + z^2 \geq 1/y\}$.

2. (a) Define the concept of a metric space.
   (b) Let $X$ be a metric space with distance function $d$, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of $X$. What is meant by saying that this sequence converges to some point $p$ of $X$?
   (c) Let $X$ and $Y$ be metric spaces with distance functions $d_X$ and $d_Y$ respectively. What is meant by saying that a function $\varphi : X \to Y$ from $X$ to $Y$ is continuous?
   (d) Let $x_1, x_2, x_3, \ldots$ be a sequence of points of the metric space $X$ which converges to some point $p$ of $X$, and let $\varphi : X \to Y$ be a continuous function from $X$ to some metric space $Y$. Prove that the sequence $\varphi(x_1), \varphi(x_2), \varphi(x_3), \ldots$ converges to $\varphi(p)$.
   (e) Let $X$ and $Y$ be metric spaces with distance functions $d_X$ and $d_Y$ respectively, let $\varphi : X \to Y$ be a function from $X$ to $Y$, and let $p$ be a point of $X$. Suppose that the function $\varphi$ is not continuous at $p$. Prove that there exists some $\varepsilon_0 > 0$ and a sequence $x_1, x_2, x_3, \ldots$ of points of $X$ converging to $p$ with the property that $d_Y(x_j, p) \geq \varepsilon_0$ for all natural numbers $j$.

3. Let $X$ be a metric space with distance function $d$.
   (a) Define precisely what is meant by saying that a sequence $x_1, x_2, x_3, \ldots$ of points of $X$ is a Cauchy sequence. Define the concept of a complete metric space.
   (b) Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be sequences of points of the metric space $X$. Suppose that the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence and that $d(x_j, y_j) \to 0$ as $j \to +\infty$. Prove that the sequence $y_1, y_2, y_3, \ldots$ is also a Cauchy sequence.
State and prove the *Contraction Mapping Theorem.*

4. (a) Define the concept of a *topological space.* What is meant by saying that a topological space $X$ is Hausdorff?

(b) What is meant by saying that a sequence $x_1, x_2, x_3, \ldots$ of points in a topological space $X$ converges to some point $p$ of $X$? What is meant by saying that a function $\varphi: X \rightarrow Y$ from a topological space $X$ to a topological space $Y$ is *continuous*?

(c) Let $X$ be a topological space, and let $A$ be a subset of $X$. Prove that $A$ is a topological space with respect to the *subspace topology* on $A$, whose open sets are subsets of $A$ of the form $A \cap V$, where $V$ is open in $X$. Prove that if $X$ is a Hausdorff space then $A$, with the subspace topology, is also a Hausdorff space.

(d) Let $X$ be an arbitrary set. Let us refer to a subset $V$ of $X$ as an *open set* if either $V = X$ or else the complement $X \setminus V$ of $V$ is finite. Prove that $X$, with these open sets, is a topological space. Show also that $X$ is Hausdorff if and only if $X$ is finite.

[Additional questions were contributed by Dr. Donal P. O’Donovan on the material that he taught.]

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