Course 212: Academic Year 1990-1991
Section 3: Topological Spaces

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Contents

3 Topological Spaces

3.1 Topologies on Sets

Definition A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set X are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all open sets in a topological space X is referred to as a *topology* on the set X.

Remark If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

Any metric space may be regarded as a topological space, where the open sets of the metric space are defined as in 1. Proposition 1.7 shows that the topological space axioms are satisfied by the collection of open sets in any metric space.

In particular, we can regard n-dimensional Euclidean space $\mathbb{R}^n$ as a topological space whose open sets are those subsets V of $\mathbb{R}^n$ with the property that, given any point $\mathbf{v}$ of V, there exists some $\delta > 0$ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{v}| < \delta\} \subset V.$$

This topology on $\mathbb{R}^n$ is referred to as the *usual topology* on $\mathbb{R}^n$. One defines the usual topologies on $\mathbb{R}$ and $\mathbb{C}$ in an analogous fashion.

Example Given any set X, one can define a topology on X where every subset of X is an open set. This topology on X is referred to as the *discrete topology* on X.

Example Given any set X, one can define a topology on X in which the only open sets are the empty set $\emptyset$ and the whole set X.

Example The *Zariski topology* on the set $\mathbb{R}$ of real numbers is defined as follows: a subset U of $\mathbb{R}$ is open (with respect to the Zariski topology) if and only if either $U = \emptyset$ or else $\mathbb{R} \setminus U$ is finite. It is a straightforward exercise to verify that the topological space axioms are satisfied, so that the set $\mathbb{R}$ of real numbers is a topological space with respect to this Zariski topology.

Definition Let X be a topological space. A subset F of X is said to be a *closed set* if and only if its complement $X \setminus F$ is an open set.

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets). The following result therefore follows directly from the relevant definitions.

Proposition 3.1 *Let X be a topological space. Then the collection of closed sets of X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are closed sets,

(ii) the intersection of any collection of closed sets is itself a closed set,

(iii) the union of any finite collection of closed sets is itself a closed set.

3.2 Subspace Topologies

Let X be a topological space with topology τ, and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $V \cap A$ for $V \in \tau$. Then τ_A is a topology on the set A. (It is a straightforward exercise to verify that the topological space axioms are satisfied by τ_A.) The topology τ_A on A is referred to as the *subspace topology* on A.

Example Let H_+ and H_- be the subsets of $\mathbb{R}^3$ defined by

$$H_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}, \qquad H_- = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}.$$

Then H_+ and H_- are open sets in $\mathbb{R}^3$. Let S^2 be the unit sphere in $\mathbb{R}^3$, defined by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and let

$$A_+ = \{(x,y,z) \in \mathbb{R}^3 : z > 0 \text{ and } x^2+y^2+z^2 = 1\},$$
$$A_- = \{(x,y,z) \in \mathbb{R}^3 : z < 0 \text{ and } x^2+y^2+z^2 = 1\}.$$

Then the sets A_+ and A_- are open in S^2 (with respect to the subspace topology on S^2), since $A_+ = H_+ \cap S^2$ and $A_- = H_- \cap S^2$. (Note that A_+ and A_- are not open in $\mathbb{R}^3$.) The complements

$$\{(x,y,z) \in \mathbb{R}^3 : z \leq 0 \text{ and } x^2+y^2+z^2 = 1\},$$

$$\{(x,y,z) \in \mathbb{R}^3 : z \geq 0 \text{ and } x^2+y^2+z^2 = 1\}$$

of A_+ and A_- in S^2 are then closed subsets of S^2.

Lemma 3.2 *Let X be a metric space with distance function d, and let A be a subset of X. A subset W of A is open with respect to the subspace topology on A if and only if, given any point w of W, there exists some $\delta > 0$ such that*

$$\{a \in A : d(a,w) < \delta\} \subset W.$$

Thus the subspace topology on A concides with the topology on A obtained on regarding A as a metric space (with respect to the distance function d).

Proof Suppose that W is open with respect to the subspace topology on A. Then there exists some open set U in X such that $W = U \cap A$. Let w be a point of W. Then there exists some $\delta > 0$ such that

$$\{x \in X : d(x,w) < \delta\} \subset U.$$

But then

$$\{a \in A : d(a,w) < \delta\} \subset U \cap A = W.$$

Conversely, suppose that W is a subset of A with the property that, for any point w of W, there exists some $\delta_w > 0$ such that

$$\{a \in A : d(a,w) < \delta_w\} \subset W.$$

Define U to be the union of the open balls $B_X(w, \delta_w)$ as w ranges over all points of W, where

$$B_X(w, \delta_w) = \{x \in X : d(x,w) < \delta_w\}.$$

The set U is an open set in X, since each open ball $B_X(w, \delta_w)$ is an open set in X (by Lemma 1.5), and any union of open sets is itself an open set. Moreover

$$B_X(w, \delta_w) \cap A = \{a \in A : d(a,w) < \delta_w\} \subset W$$

for any $w \in W$. Therefore $U \cap A \subset W$. However $W \subset U \cap A$, since, $W \subset A$ and $\{w\} \subset B_X(w, \delta_w) \subset U$ for any $w \in W$. Thus $W = U \cap A$, where U is an open set in X. We deduce that W is open with respect to the subspace topology on A. ∎

Example Let X be any subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then the subspace topology on X coincides with the topology on X generated by the Euclidean distance function on X. We refer to this topology as the *usual topology* on X.

Let X be a topological space, and let A be a subset of X. One can readily verify the following:—

> a subset B of A is closed in A (relative to the subspace topology on A) if and only if $B = A \cap F$ for some closed subset F of X,
>
> if A is itself open in X then a subset B of A is open in A if and only if it is open in X,
>
> if A is itself closed in X then a subset B of A is closed in A if and only if it is closed in X.

3.3 Hausdorff Spaces

Definition A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

> if x and y are distinct points of X then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).

Lemma 3.3 *All metric spaces are Hausdorff spaces*

Proof Let X be a metric space with distance function d, and let x and y be points of X, where $x \neq y$. Let $U = B_X(x, \varepsilon)$ and $V = B_X(y, \varepsilon)$, where $\varepsilon = \frac{1}{2}d(x, y)$. (Here $B_X(x, \varepsilon)$ and $B_X(y, \varepsilon)$ denote the open balls of radius ε about the points x and y respectively.) Now U and V are open sets, by Lemma 1.5. Also $U \cap V = \emptyset$. To see this we note that, were $U \cap V$ non-empty, then there would exist a point z in X such that $d(x, z) < \varepsilon$ and $d(z, y) < \varepsilon$. But this is impossible, since it would then follow from the Triangle Inequality that $d(x, y) < 2\varepsilon$, contrary to the choice of ε. We deduce that $U \cap V = \emptyset$, as required. ∎

We now give an example of a topological space which is not a Hausdorff space.

Example Consider the set $\mathbb{R}$ of real numbers with the Zariski topology, in which a subset of $\mathbb{R}$ is open if and only if it is empty or its complement is a finite set. The intersection of any two non-empty open sets is always non-empty. (Indeed if U and V are non-empty open sets then $U = \mathbb{R} \setminus F_1$ and $V = \mathbb{R} \setminus F_2$, where F_1 and F_2 are finite sets of real numbers. But then $U \cap V = \mathbb{R} \setminus (F_1 \cup F_2)$, which is non-empty, since $\mathbb{R}$ is infinite but $F_1 \cup F_2$ is finite.) It follows easily from this that the Zariski topology on $\mathbb{R}$ does not satisfy the Hausdorff axiom.

3.4 Continuous Functions between Topological Spaces

Definition Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be *continuous* if and only if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) \equiv \{x \in X : f(x) \in V\}.$$

A continuous function from X to Y is often referred to as a *map* from X to Y.

Lemma 3.4 *Let X, Y and Z be topological spaces, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Then the composition $g \circ f\colon X \to Z$ of the functions f and g is continuous.*

Proof Let V be an open set in Z. Then $g^{-1}(V)$ is open in Y (since g is continuous), and hence $f^{-1}(g^{-1}(V))$ is open in X (since f is continuous). But $f^{-1}(g^{-1}(V) = (g \circ f)^{-1}(V)$. Thus the composition function $g \circ f$ is continuous. ∎

Lemma 3.5 *Let X and Y be topological spaces, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(G)$ is closed in X for every closed subset G of Y.*

Proof This result follows easily from the observation that $X \setminus f^{-1}(G) = f^{-1}(Y \setminus G)$ for all subsets G of Y. ∎

Definition Let X and Y be topological spaces. A function $h\colon X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

(i) the function $h: X \to Y$ is both injective and surjective (so that the function $h: X \to Y$ has a well-defined inverse $h^{-1}: Y \to X$),

(ii) the function $h: X \to Y$ and its inverse $h^{-1}: Y \to X$ are both continuous.

If there exists a homeomorphism $h: X \to Y$ from the topological space X to the topological space Y then the topological spaces X and Y are said to be *homeomorphic*.

If $h: X \to Y$ is a homeomorphism between topological spaces X and Y then h induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being identical as topological spaces.

3.5 Sequences and Convergence

Definition Let $x_1, x_2, x_3, \ldots$ be a sequence of points in a topological space X. Let l be a point of X. The sequence (x_j) is said to *converge* to l if and only if, given any open set U containing the point l, there exists some natural number N such that $x_j \in U$ for all j satisfying $j \geq N$. If the sequence (x_j) converges to l then we refer to l as a *limit* of the sequence.

This definition of convergence generalizes the definition of convergence for a sequence of points in a metric space (see Lemma 1.8).

It can happen that a sequence of points in a topological space can have more than one limit. For example, consider the set $\mathbb{R}$ of real numbers with the Zariski topology. (The open sets of $\mathbb{R}$ in the Zariski topology are the empty set and those subsets of $\mathbb{R}$ whose complements are finite.) Let $x_1, x_2, x_3, \ldots$ be the sequence in $\mathbb{R}$ defined by $x_j = j$ for all natural numbers j. One can readily check that, for any real number l, the sequence (x_j) converges to l (with respect to the Zariski topology on $\mathbb{R}$).

Lemma 3.6 *A sequence $x_1, x_2, x_3, \ldots$ of points in a Hausdorff space X converges to at most one limit.*

Proof Suppose that l and m were limits of the sequence (x_j), where $l \neq m$. Then there would exist open sets U and V such that $l \in U$, $m \in V$ and $U \cap V = \emptyset$, since X is a Hausdorff space. But then there would exist natural numbers N_1 and N_2 such that $x_j \in U$ for all j satisfying $j \geq N_1$ and $x_j \in V$ for all j satisfying $j \geq N_2$. But then $x_j \in U \cap V$ for all j satisfying $j \geq N_1$ and $j \geq N_2$, which is impossible, since $U \cap V = \emptyset$. This contradiction shows that the sequence (x_j) has at most one limit. ∎

Lemma 3.7 *Let X be a topological space, and let F be a closed set in X. Let $(x_j : j \in \mathbb{N})$ be a sequence of points in F. Suppose that the sequence (x_j) converges to some point l of X. Then $l \in F$.*

Proof Suppose that l were a point belonging to the complement $X \setminus F$ of F. Now $X \setminus F$ is open (since F is closed). Therefore there would exist some natural number N such that $x_j \in X \setminus F$ for all values of j satisfying $j \geq N$, contradicting the fact that $x_j \in F$ for all j. This contradiction shows that l must belong to F, as required. ∎

Lemma 3.8 *Let X and Y be topological spaces, and let $f: X \to Y$ be a continuous function from X to Y. Let $x_1, x_2, x_3, \ldots$ be a sequence of points in X which converges to some point l of X. Then the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(l)$.*

Proof Let V be an open set in Y which contains the point $f(l)$. Then $f^{-1}(V)$ is an open set in X which contains the point l. It follows that there exists some natural number N such that $x_j \in f^{-1}(V)$ whenever $j \geq N$. But then $f(x_j) \in V$ whenever $j \geq N$. We deduce that the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(l)$, as required. ∎

3.6 Neighbourhoods, Closures and Interiors

Definition Let X be a topological space, and let x be a point of X. Let N be a subset of X which contains the point x. Then N is said to be a *neighbourhood* of the point x if and only if there exists an open set U for which $x \in U$ and $U \subset N$.

One can readily verify that this definition of neighbourhoods for points in a topological space is consistent with that given in 1 in the context of points in a metric space.

Example Let A be the closed unit ball in $\mathbb{R}^3$, defined by

$$A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\},$$

and let B be the open unit ball, defined by

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}.$$

If $\mathbf{b}$ is any point of B then A is a neighbourhood of $\mathbf{b}$, since B is an open set for which $\mathbf{b} \in B$ and $B \subset A$. If however $\mathbf{b}$ is a point on the unit sphere S^2, where

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

then A is not a neighbourhood of $\mathbf{b}$ since any open set containing $\mathbf{b}$ must also contain points whose distance from the origin exceeds 1.

Lemma 3.9 *Let X be a topological space. A subset V of X is open in X if and only if V is a neighbourhood of each point belonging to V.*

Proof It follows directly from the definition of neighbourhoods that an open set V is a neighbourhood of any point belonging to V. Conversely, suppose that V is a subset of X with the property that V is a neighbourhood of each point belonging to V. Then, given any point v of V, there exists an open set U_v such that $v \in U_v$ and $U_v \subset V$. Thus V is the union of the open sets U_v as v ranges over all points of V. But any set that is the union of open sets is itself open. Thus V is an open set in X. ∎

Definition Let X be a topological space and let A be a subset of X. The *closure* $\overline{A}$ of A in X is defined to be the intersection of all of the closed subsets of X that contain A. The *interior* A^0 of A in X is defined to be the union of all of the open subsets of X that are contained in A.

Proposition 3.10 *Let X be a topological space and let A be a subset of X. Then the closure $\overline{A}$ of A is uniquely chararcterized by the following two properties:*

(i) $\overline{A}$ is a closed set satisfying $A \subset \overline{A}$,

(ii) $\overline{A} \subset F$ for any closed set F satisfying $A \subset F$.

Similarly the interior A^0 of A is uniquely characterized by the following two properties:

(i) A^0 is an open set satisfying $A^0 \subset A$,

(ii) $U \subset A^0$ for any open set U satisfying $U \subset A$.

Proof Note that the closure $\overline{A}$ of A contains A since it is the intersection of some non-empty collection of subsets of X each of which contains the set A. (One of these subsets is X itself.) Moreover the closure of A is a closed set, since any intersection of closed sets is closed. It follows directly from the definition of the closure $\overline{A}$ of A that if F is a closed subset of X containing A then $\overline{A} \subset F$.

The interior A^0 is an open set contained in A since it is the union of a collection of open sets contained in A, and any union of open sets is itself an open set. It follows directly from the definition of the interior A^0 of A that if U is an open subset of X which is contained in A then $U \subset A^0$. ∎

The following result follows directly from the relevant definitions.

Lemma 3.11 *Let X be a topological space, let A be a subset of X, and let x be a point of A. Then A is a neighbourhood of x if and only if x belongs to the interior A^0 of A.*

Lemma 3.12 *Let X be a topological space, and let A be a subset of X. Suppose that a sequence $x_1, x_2, x_3, \ldots$ of points of A converges to some point l of X. Then $l \in \overline{A}$.*

Proof This follows directly from Lemma 3.7, since $\overline{A}$ is closed and $A \subset \overline{A}$. ∎

Example Let S^2 be the unit sphere in $\mathbb{R}^3$, and let A and C denote the subsets of S^2 defined by

$$\begin{aligned} A &= \{(x, y, z) \in \mathbb{R}^3 : z > 0 \text{ and } x^2 + y^2 + z^2 = 1\}, \\ C &= \{(x, y, z) \in \mathbb{R}^3 : z \geq 0 \text{ and } x^2 + y^2 + z^2 = 1\}. \end{aligned}$$

We claim that $\overline{A} = C$. Now C is a closed set in S^2. Therefore $\overline{A} \subset C$. We now show that $C \setminus A \subset \overline{A}$. Let $\mathbf{u}$ be any point of $C \setminus A$. Then $\mathbf{u} = (x, y, 0)$, where x and y are real numbers satisfying $x^2 + y^2 = 1$. For each natural number n define

$$\mathbf{v}_n = (x \cos(1/n), y \cos(1/n), \sin(1/n)).$$

Then $\mathbf{v}_n \to \mathbf{u}$ as $n \to +\infty$. But $\mathbf{v}_n$ is a point of A for all n. It follows from Lemma 3.12 that $\mathbf{u} \in \overline{A}$. This shows that $C \setminus A \subset \overline{A}$, and hence $C \subset \overline{A}$ (since $A \subset \overline{A}$). We conclude that $C = \overline{A}$.

3.7 Product Topologies

Let $X_1, X_2, \ldots, X_n$ be sets. The *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ of the sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, 2, \ldots, n$. If $A_1, A_2, \ldots, A_n$ are subsets of $X_1, X_2, \ldots, X_n$ respectively then can regard the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ as a subset of $X_1 \times X_2 \times \cdots \times X_n$.

Now let $X_1, X_2, \ldots X_n$ be topological spaces. We define a topology on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ whose open sets are characterized by the following property:—

> a subset U of $X_1 \times X_2 \times \cdots \times X_n$ is open if and only if, for each point $(u_1, u_2, \ldots, u_n)$ of U, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $u_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset U$.

One can readily verify that the topological space axioms are satisfied: the empty set $\emptyset$ and the whole space $X_1 \times X_2 \times \cdots \times X_n$ are open sets, any union of open sets is open, and any finite intersection of open sets is open. This topology is referred to as the *product topology* on $X_1 \times X_2 \times \cdots \times X_n$.

Note that if V_i is an open set in X_i for $i = 1, 2, \ldots, n$ then $V_1 \times V_2 \times \cdots \times V_n$ is an open set in $X_1 \times X_2 \times \cdots \times X_n$.

Lemma 3.13 *Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let p_i denote the function from $X_1 \times X_2 \times \cdots \times X_n$ to X_i defined by*

$$p_i(x_1, x_2, \ldots, x_n) = x_i$$

for $i = 1, 2, \ldots, n$. Then $p_1, p_2, \ldots, p_n$ are continuous functions.

Proof Let V be an open set in X_i. Then

$$p_i^{-1}(V) = X_1 \times \cdots \times X_{i-1} \times V \times X_{i+1} \times \cdots \times X_n.$$

Thus $p_i^{-1}(V)$ is the Cartesian product of open subsets of $X_1, X_2, \ldots, X_n$. It follows from the definition of the product topology that $p_i^{-1}(V)$ is an open set in $X_1 \times X_2 \times \cdots \times X_n$. Thus $p_i \colon X_1 \times X_2 \times \cdots \times X_n \to X_i$ is continuous. ∎

Theorem 3.14 *Let $X_1 \times X_2 \times \cdots \times X_n$ be the Cartesian product of the topological spaces $X_1, X_2, \ldots, X_n$. Let Z be a topological space. A function $f \colon Z \to X_1 \times X_2 \times \cdots \times X_n$ is continuous if and only if $p_i \circ f \colon Z \to X_i$ is continuous for $i = 1, 2, \ldots, n$, where $p_i \colon X_1 \times X_2 \times \cdots \times X_n \to X_i$ denotes the projection function sending $(x_1, x_2, \ldots, x_n)$ to x_i.*

Proof Suppose that $f \colon Z \to X_1 \times X_2 \times \cdots \times X_n$ is continuous. Then $p_i \circ f$ is continuous for all i, since p_i is continuous (Lemma 3.13) and a composition of continuous functions is continuous (Lemma 3.4).

Conversely suppose that $p_i \circ f$ is continuous for all i. Let U be an open set in $X_1 \times X_2 \times \cdots \times X_n$. We must show that $f^{-1}(U)$ is open in Z. To show this, we prove that, for each $z \in f^{-1}(U)$, there exists some open set N_z in Z such that $z \in N_z$ and $N_z \subset f^{-1}(U)$.

Let z be a point of $f^{-1}(U)$, and let $f(z) = (u_1, u_2, \ldots, u_n)$. Now U is open in $X_1 \times X_2 \times \cdots \times X_n$. It follows from the definition of the product topology that there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ respectively such that $u_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset U$. Let N_z be the subset of Z defined by

$$N_z = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \cap \cdots \cap f_n^{-1}(V_n).$$

Now $f_i^{-1}(V_i)$ is an open subset of Z for $i = 1, 2, \ldots, n$, since V_i is open in X_i and $f_i \colon Z \to X_i$ is continuous. But any finite intersection of open sets is open. Thus N_z is open in Z. Moreover

$$f(N_z) \subset V_1 \times V_2 \times \cdots \times V_n \subset U,$$

so that $N_z \subset f^{-1}(U)$.

Now $f^{-1}(U)$ is the union of the open sets N_z as z ranges over all points of $f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in Z. This shows that $f \colon Z \to X_1 \times X_2 \times \cdots \times X_n$ is continuous. ▮

Proposition 3.15 *The usual topology on $\mathbb{R}^n$ (induced by the Euclidean distance function on $\mathbb{R}^n$) coincides with the product topology on $\mathbb{R}^n$ obtained on regarding $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ of n copies of the real line $\mathbb{R}$.*

Proof We must show that a subset U of $\mathbb{R}^n$ is open with respect to the usual topology if and only if it is open with respect to the product topology.

Let U be a subset of $\mathbb{R}^n$ that is open with respect to the usual topology, and let $\mathbf{u}$ be a point of U, where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. It follows from the definition of the usual topology on $\mathbb{R}^n$ that there exists some $\delta > 0$ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

Let $I_1, I_2, \ldots, I_n$ be the open intervals in $\mathbb{R}$ defined by

$$I_i = \{t \in \mathbb{R} : u_i - \frac{\delta}{\sqrt{n}} < t < u_i + \frac{\delta}{\sqrt{n}}\} \qquad (i = 1, 2, \ldots, n).$$

Then $I_1, I_2, \ldots, I_n$ are open sets in $\mathbb{R}$ containing $(u_1, u_2, \ldots, u_n)$, and $I_1 \times I_2 \times \cdots \times I_n \subset U$. Indeed if $\mathbf{x}$ is a point of $I_1 \times I_2 \times \cdots \times I_n$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then $|x_i - u_i| < \delta/\sqrt{n}$ for all i, and hence $|\mathbf{x} - \mathbf{u}| < \delta$. This shows that any subset U of $\mathbb{R}^n$ that is open with respect to the usual topology on $\mathbb{R}^n$ is also open with respect to the product topology on $\mathbb{R}^n$.

Conversely suppose that U is a subset of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$, and let $\mathbf{u}$ be a point of U, where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. Then there exist open sets $V_1, V_2, \ldots, V_n$ in $\mathbb{R}$ containing $u_1, u_2, \ldots, u_n$ respectively such that $V_1 \times V_2 \times \cdots \times V_n \subset U$. Now we can find $\delta_1, \delta_2, \ldots, \delta_n$ such that $\delta_i > 0$ and $(u_i - \delta_i, u_i + \delta_i) \subset V_i)$ for all i. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \ldots, \delta_n$. Then $\delta > 0$. If $\mathbf{x}$ is a point of $\mathbb{R}^n$ satisfying $|\mathbf{x} - \mathbf{u}| < \delta$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then $|x_i - u_i| < \delta_i$ and hence $x_i \in V_i$ for $i = 1, 2, \ldots, n$. It follows that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U.$$

This shows that any subset U of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$ is also open with respect to the usual topology on $\mathbb{R}^n$. ∎

The following result is the analogue of Proposition 1.17 for functions from a topological space X to $\mathbb{R}^n$.

Corollary 3.16 *Let X be a topological space and let $f: X \to \mathbb{R}^n$ be a function from X to $\mathbb{R}^n$. Let us write*

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$, where the components $f_1, f_2, \ldots, f_n$ of f are functions from X to $\mathbb{R}$. The function f is continuous if and only if its components $f_1, f_2, \ldots, f_n$ are all continuous.

Proof Note that $f_i = p_i \circ f$ for $i = 1, 2, \ldots, n$, where $p_i: \mathbb{R}^n \to \mathbb{R}$ denotes the projection function mapping $(x_1, x_2, \ldots, x_n)$ to x_i. The desired result now follows immediately from Proposition 3.15 and Theorem 3.14. ∎

Lemma 3.17 *Let $X_1, X_2, \ldots, X_n$ be Hausdorff spaces. Then the space $X_1 \times X_2 \times \ldots, X_n$ is Hausdorff.*

Proof Let $X = X_1 \times X_2 \times \ldots, X_n$, and let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be distinct points of X. Then $x_i \neq y_i$ for some integer i between 1 and n. But then there exists open sets U and V in X_i such that $x_i \in U$, $y_i \in V$ and $U \cap V = \emptyset$ (since X_i is a Hausdorff space). Let $p_i: X \to X_i$ denote the projection function. Then $p_i^{-1}(U)$ and $p_i^{-1}(V)$ are open sets in X, since p_i is continuous. Moreover $(x_1, x_2, \ldots, x_n)$ belongs to $p_i^{-1}(U)$, $(y_1, y_2, \ldots, y_n)$ belongs to $p_i^{-1}(V)$ and $p_i^{-1}(U) \cap p_i^{-1}(V) = \emptyset$. This shows that X is Hausdorff, as required. ∎

3.8 Identification Maps and Quotient Topologies

Definition Let X and Y be topological spaces and let $q: X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:

(i) the function $q: X \to Y$ is surjective,

(ii) a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

It follows directly from condition (ii) in the definition of an identification map that any identification map is continuous.

Lemma 3.18 *Let X be a topological space, let Y be a set, and let $q: X \to Y$ be a surjection. Then there is a unique topology on Y for which the function $q: X \to Y$ is an identification map.*

Proof Let τ be the collection consisting of all subsets U of Y for which $q^{-1}(U)$ is open in X. Now $q^{-1}(\emptyset) = \emptyset$, and $q^{-1}(Y) = X$, so that $\emptyset \in \tau$ and $Y \in \tau$. It is readily verified that any union of sets belonging to τ is itself a set belonging to τ, and that any finite intersection of sets belonging to τ is itself a set belonging to τ. Thus τ is a topology on Y, and the function $q: X \to Y$ is an identification map with respect to the topology τ on Y. Clearly the topology τ is the unique topology on Y for which the function $q: X \to Y$ is a identification map. ∎

Let X be a topological space, let Y be a set, and let $q: X \to Y$ be a surjection. The unique topology on Y for which the function q is an identifcation map is referred to as the *quotient topology* (or *identification topology*) on Y.

Lemma 3.19 *Let X and Y be topological spaces and let $q: X \to Y$ be an identification map. Let Z be a topological space, and let $f: Y \to Z$ be a function from Y to Z. Then the function f is continuous if and only if the composition function $f \circ q: X \to Z$ is continuous.*

Proof Suppose that f is continuous. Then the composition function $f \circ q$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $f \circ q$ is continuous. Let U be an open set in Z. Then $q^{-1}(f^{-1}(U))$ is open in X (since $f \circ q$ is continuous), and hence $f^{-1}(U)$ is open in Y (since the function q is an identification map). Therefore the function f is continuous, as required. ∎

Example Let S^n be the n-sphere, consisting of all points $\mathbf{x}$ in $\mathbb{R}^{n+1}$ satisfying $|\mathbf{x}| = 1$. Let $\mathbb{R}P^n$ be the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin (i.e., $\mathbb{R}P^n$ is the set of all one-dimensional vector subspaces of $\mathbb{R}^{n+1}$). Let $q: S^n \to \mathbb{R}P^n$ denote the function which sends a point $\mathbf{x}$ of S^n to the element of $\mathbb{R}P^n$ represented by the line in $\mathbb{R}^{n+1}$ that passes through both $\mathbf{x}$ and the origin. Note that each point of $\mathbb{R}P^n$ is the image (under q) of exactly two points on S^n, if $\mathbf{x} \in S^n$ is one of these points then the other is $-\mathbf{x}$. The function q induces a corresponding quotient topology (or identification topology) on $\mathbb{R}P^n$. The topological space $\mathbb{R}P^n$ is referred to as *real projective n-space*. In particular $\mathbb{R}P^2$ is referred to as the *real projective plane*. It follows from Lemma 3.19 that a function $f: \mathbb{R}P^n \to Z$ from $\mathbb{R}P^n$ to any topological space Z is continuous if and only if the composition function $f \circ q$ is continuous.