Course 212: Academic Year 1990-1991
Section 2: Cauchy Sequences and Completeness

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2 Cauchy Sequences and Completeness

2.1 Cauchy Sequences

Definition Let $X$ be a metric space with distance function $d$. A sequence $x_1, x_2, x_3, x_4, \ldots$ of points of $X$ is said to be a Cauchy sequence in $X$ if and only if, given any $\varepsilon > 0$ there exists some natural number $N$ such that $d(x_j, x_k) < \varepsilon$ for all natural numbers $j$ and $k$ satisfying $j \geq N$ and $k \geq N$.

Every convergent sequence in a metric space is a Cauchy sequence. Indeed let $X$ be a metric space with distance function $d$, and let $x_1, x_2, x_3, x_4, \ldots$ be a sequence of points in $X$ which converges to some element $l$ of $X$. Given any $\varepsilon > 0$, there exists some natural number $N$ such that $d(x_n, l) < \varepsilon/2$ whenever $n \geq N$. But then it follows from the Triangle Inequality (Axiom (iii)) that

$$d(x_j, x_k) \leq d(x_j, l) + d(l, x_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $j \geq N$ and $k \geq N$.

Definition A metric space $X$ is said to be complete if every Cauchy sequence in $X$ converges to some point of $X$.

Lemma 2.1 Let $X$ be a complete metric space, and let $A$ be a closed subset of $X$. Then $A$ is a complete metric space.

Proof Let $a_1, a_2, a_3, a_4, \ldots$ be a Cauchy sequence in $A$. This Cauchy sequence must converge to some point $l$ of $X$, since $X$ is complete. But then $l$ must be an element of $A$, by Lemma 1.11, since $A$ is closed in $X$. We deduce that $A$ is complete, as required. 

Example The spaces $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces with respect to the distance function given by $d(z, w) = |z - w|$. Indeed this result is Cauchy’s Criterion for Convergence (proved in Course 121). However the space $\mathbb{Q}$ of rational numbers (with distance function $d(q, r) = |q - r|$) is not complete. Indeed one can construct an infinite sequence $q_1, q_2, q_3, q_4, \ldots$ of rational numbers which converges (in $\mathbb{R}$) to $\sqrt{2}$. Such a sequence of rational numbers is a Cauchy sequence in both $\mathbb{R}$ and $\mathbb{Q}$. However this Cauchy sequence does not converge to an element of the metric space $\mathbb{Q}$ (since $\sqrt{2}$ is an irrational number). Thus the metric space $\mathbb{Q}$ is not complete.

Lemma 2.2 The metric space $\mathbb{R}^n$ (with the Euclidean distance function) is a complete metric space.
**Proof** Let \( x_1, x_2, x_3, x_4, \ldots \) be a Cauchy sequence in \( \mathbb{R}^n \). We must show that this sequence converges to some element \( u \) of \( \mathbb{R}^n \). Let us write \( x_j = (\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{nj}) \) for all natural numbers \( j \). Now for any value of \( m \) between 1 and \( n \) the sequence \( \alpha_{m1}, \alpha_{m2}, \alpha_{m3}, \alpha_{m4}, \ldots \) is a Cauchy sequence of real numbers. (This follows easily \( |\alpha_{mj} - \alpha_{mk}| \leq |x_j - x_k| \).) It follows from Cauchy’s Criterion for Convergence that the sequence \( \alpha_{m1}, \alpha_{m2}, \alpha_{m3}, \alpha_{m4}, \ldots \) converges to some real number \( u_m \).

Let \( u = (u_1, u_2, \ldots, u_n) \). We claim that \( x_j \to u \) as \( j \to +\infty \). Indeed let \( \varepsilon > 0 \) be given. Then there exist natural numbers \( N_1, N_2, \ldots, N_n \) such that \( |\alpha_{mj} - u_m| < \varepsilon/\sqrt{n} \) whenever \( j \geq N_m \) (where \( m = 1, 2, \ldots, n \)). Let \( N \) be the maximum of \( N_1, N_2, \ldots, N_n \). If \( j \geq N \) then

\[
|x_j - u|^2 = \sum_{m=1}^{n} (\alpha_{mj} - u_m)^2 < \varepsilon^2.
\]

Thus \( x_j \to u \) as \( j \to +\infty \), as required. □

**Example** The \( n \)-sphere \( S^n \) (with the chordal distance function given by \( d(x, y) = |x - y| \)) is a complete metric space. Indeed \( S^n \) is the closed subset of \( \mathbb{R}^{n+1} \) given by \( S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \), and a closed subset of a complete metric space is itself complete (Lemma 2.1).

Let \( X \) and \( Y \) be metric spaces, with distance functions \( d_X \) and \( d_Y \) respectively. We say that a function \( f: X \to Y \) from \( X \) to \( Y \) is bounded if and only if there exists some non-negative constant \( K \) such that \( d_Y(f(x), f(x')) \leq K \) for all points \( x \) and \( x' \) in \( X \).

It is easy to see that a function \( f: X \to \mathbb{R}^n \) is bounded if and only if there exists some non-negative constant \( K \) such that \( |f(x)| \leq K \) for all \( x \in X \).

Let \( f: X \to Y \) and \( g: X \to Y \) be bounded functions from \( X \) to \( Y \). Then the function \( x \mapsto d_Y(f(x), g(x)) \) is bounded above on \( X \). Indeed suppose that we choose some point \( x_0 \) of \( X \). There exist constants \( K_1 \) and \( K_2 \) such that \( d_Y(f(x), f(x_0)) \leq K_1 \) and \( d_Y(g(x), g(x_0)) \leq K_2 \) for all \( x \in X \), since the functions \( f \) and \( g \) are both bounded. But then

\[
d_Y(f(x), g(x)) \leq d_Y(f(x), f(x_0)) + d_Y(f(x_0), g(x_0)) + d_Y(g(x_0), g(x)) \leq d_Y(f(x_0), g(x_0)) + K_1 + K_2
\]

for all points \( x \) of \( X \), by a straightforward application of the triangle inequality. We define

\[
\rho(f, g) = \sup\{d_Y(f(x), g(x)) : x \in X\}
\]
for all bounded functions $f$ and $g$ from $X$ to $Y$. If $f$, $g$, and $h$ are bounded functions from $X$ to $Y$ then

$$d_Y(f(x), h(x)) \leq d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \leq \rho(f, g) + \rho(g, h)$$

for all $x \in X$, and hence $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$, showing that the distance function $\rho$ satisfies the Triangle Inequality.

**Theorem 2.3** Let $X$ and $Y$ be metric spaces with distance functions $d_X$ and $d_Y$ respectively. Suppose that $Y$ is complete. Then the set $C(X,Y)$ of bounded continuous functions from $X$ to $Y$ is a complete metric space with respect to the distance function $\rho$, where $\rho(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$ for all $f, g \in C(X,Y)$.

**Proof** One can readily verify that the metric space axioms are satisfied by the distance function $\rho$ on $C(X,Y)$.

Let $f_1, f_2, f_3, f_4, \ldots$ be a Cauchy sequence in $C(X,Y)$. Choose any $x \in X$. Now $d_Y(f_j(x), f_k(x)) \leq \rho(f_j, f_k)$ for all $j$ and $k$. Using this inequality, one can easily deduce that $f_1(x), f_2(x), f_3(x), f_4(x), \ldots$ is a Cauchy sequence in $Y$. But every Cauchy sequence in $Y$ is convergent, since $Y$ is complete. Thus there exists some point $f(x)$ of $Y$ such that $f_n(x) \to f(x)$ as $n \to +\infty$. In this way we define a function $f: X \to Y$ sending any point $x$ of $X$ to the point $f(x)$ defined as above.

Let $\varepsilon > 0$ be given. Then there exists some natural number $N$ such that $\rho(f_j, f_k) < \varepsilon/3$ whenever $j \geq N$ and $k \geq N$, since the sequence $f_1, f_2, f_3, f_4, \ldots$ is a Cauchy sequence in $C(X,Y)$. Let $x$ be any point of $X$. Then $d_Y(f_j(x), f_k(x)) < \varepsilon/3$ whenever $j \geq N$ and $k \geq N$. But $d_Y(f_j(x), f_k(x))$ converges to $d_Y(f_j(x), f(x))$ as $k \to +\infty$, by Lemma 1.2. We deduce that, given any $\varepsilon > 0$, there exists some natural number $N$ such that $d_Y(f_j(x), f(x)) \leq \varepsilon/3$ for all $x \in X$ and $j$ satisfying $j \geq N$. In particular

$$d_Y(f(x), f(x')) \leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(x')) + d_Y(f_N(x'), f(x')) \leq d_Y(f_N(x), f_N(x')) + \frac{2\varepsilon}{3}.$$ 

Now the function $f_N$ is bounded. Therefore there exists some non-negative constant $K$ such that $d_Y(f_N(x), f_N(x')) \leq K$ for all $x, x' \in X$. But then

$$d_Y(f(x), f(x')) \leq K + 2\varepsilon/3$$

for all $x, x' \in X$, showing that $f$ is bounded. Also $f_N$ is continuous at $x$, and hence there exists some $\delta > 0$ such that $d_Y(f_N(x), f_N(x')) < \varepsilon/3$ whenever
Let \( X \) be a complete metric space, and let \( \lambda \) be a function from \( X \) to itself. The function \( T \) is said to be a contraction mapping if there exists some constant \( \lambda \) satisfying \( 0 \leq \lambda < 1 \) such that \( d(T(x), T(x')) \leq \lambda d(x, x') \) for all \( x, x' \in X \).

One can readily check that any contraction map \( T: X \to X \) on a metric space \((X, d)\) is continuous. Indeed let \( x \) be a point of \( X \), and let \( \varepsilon > 0 \) be given. Then \( d(T(x), T(x')) < \varepsilon \) for all points \( x' \) of \( X \) satisfying \( d(x, x') < \varepsilon \).

**Theorem 2.5 (Contraction Mapping Theorem)** Let \( X \) be a complete metric space, and let \( T: X \to X \) be a contraction mapping defined on \( X \). Then \( T \) has a unique fixed point in \( X \) (i.e., there exists a unique point \( x \) of \( X \) for which \( T(x) = x \)).

**Proof** Let \( \lambda \) be chosen such that \( 0 \leq \lambda < 1 \) and \( d(T(u), T(u')) \leq \lambda d(u, u') \) for all \( u, u' \in X \), where \( d \) is the distance function on \( X \). First we show the existence of the fixed point \( x \). Let \( x_0 \) be any point of \( X \), and define a sequence \( x_0, x_1, x_2, x_3, x_4, \ldots \) of points of \( X \) by the condition that \( x_n = T(x_{n-1}) \) for all natural numbers \( n \). It follows by induction on \( n \) that \( d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0) \). Using the Triangle Inequality, we deduce that if \( j \) and \( k \) are natural numbers satisfying \( k > j \) then

\[
d(x_k, x_j) \leq \sum_{n=j}^{k-1} d(x_{n+1}, x_n) \leq \frac{\lambda^j - \lambda^k}{1 - \lambda} d(x_1, x_0) \leq \frac{\lambda^j}{1 - \lambda} d(x_1, x_0).
\]

Let \( F \) be a closed subset of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Then \( F \) is a complete metric space, by Lemma 2.1 and Lemma 2.2. The following result therefore follows immediately from Theorem 2.3.

**Corollary 2.4** Let \( X \) be a metric space and let \( F \) be a closed subset of \( \mathbb{R}^n \). Then the space \( C(X, F) \) of bounded continuous functions from \( X \) to \( F \) is a complete metric space with respect to the distance function \( \rho \), where \( \rho(f, g) = \sup_{x \in X} |f(x) - g(x)| \) for all \( f, g \in C(X, \mathbb{R}^n) \).

### 2.2 The Contraction Mapping Theorem

**Definition** Let \( X \) be a metric space with distance function \( d \), and let \( T: X \to X \) be a function from \( X \) to itself. The function \( T \) is said to be a contraction mapping if there exists some constant \( \lambda \) satisfying \( 0 \leq \lambda < 1 \) such that \( d(T(x), T(x')) \leq \lambda d(x, x') \) for all \( x, x' \in X \).
(Here we have used the identity
\[ \lambda^j + \lambda^{j+1} + \cdots + \lambda^{k-1} = \frac{\lambda^j - \lambda^k}{1 - \lambda}. \]

Using the fact that 0 ≤ λ < 1, we deduce that the sequence \((x_n)\) is a Cauchy sequence in \(X\). This Cauchy sequence must converge to some point \(x\) of \(X\), since \(X\) is complete. But then, using Lemma 1.4, we see that
\[ T(x) = T\left(\lim_{n \to +\infty} x_n\right) = \lim_{n \to +\infty} T(x_n) = \lim_{n \to +\infty} x_{n+1} = x, \]
so that \(x\) is a fixed point of \(T\).

If \(x'\) were another fixed point of \(T\) then we would have
\[ d(x', x) = d(T(x'), T(x)) \leq \lambda d(x', x). \]
But this is impossible unless \(x' = x\), since \(\lambda < 1\). Thus the fixed point \(x\) of the contraction map \(T\) is unique. 

### 2.3 Picard’s Theorem

We now use a number of the results of this section in order to prove an existence theorem for solutions of ordinary differential equations known as Picard’s Theorem.

**Theorem 2.6 (Picard’s Theorem)** Let \(F: U \to \mathbb{R}\) be a continuous function defined over some open set \(U\) in the plane \(\mathbb{R}^2\), and let \((x_0, t_0)\) be an element of \(U\). Suppose that there exists some non-negative constant \(M\) such that
\[ |F(u, t) - F(v, t)| \leq M|u - v| \text{ for all } (u, t) \in U \text{ and } (v, t) \in U. \]
Then there exists a continuous function \(\varphi: [t_0 - \delta, t_0 + \delta] \to \mathbb{R}\) defined on the interval \([t_0 - \delta, t_0 + \delta]\) for some \(\delta > 0\) such that \(x = \varphi(t)\) is a solution to the differential equation
\[ \frac{dx(t)}{dt} = F(x(t), t) \]
with initial condition \(x(t_0) = x_0\).

**Proof** Solving the differential equation with the initial condition \(x(t_0) = x_0\) is equivalent to finding a continuous function \(\varphi: I \to \mathbb{R}\) satisfying the integral equation
\[ \varphi(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s) \, ds. \]
where \( I \) denotes the closed interval \([t_0 - \delta, t_0 + \delta]\). (Note that any continuous function \( \varphi \) satisfying this integral equation is automatically differentiable, since the indefinite integral of a continuous function is always differentiable.)

Let \( K = |F(x_0, t_0)| + 1 \). Using the continuity of the function \( F \), together with the fact that \( U \) is open in \( \mathbb{R}^2 \), one can find some \( \delta_0 > 0 \) such that the open disk of radius \( \delta_0 \) about \((x_0, t_0)\) is contained in \( U \) and \( |F(x, t)| \leq K \) for all points \((x, t)\) in this open disk. Now choose \( \delta > 0 \) such that

\[
\delta \sqrt{1 + K^2} < \delta_0 \quad \text{and} \quad M \delta < 1.
\]

Note that if \(|t - t_0| \leq \delta \) and \(|x - x_0| \leq K \delta \) then \((x, t)\) belongs to the open disk of radius \( \delta_0 \) about \((x_0, t_0)\), and hence \((x, t) \in U \) and \(|F(x, t)| \leq K \).

Let \( J \) denote the closed interval \([x_0 - K \delta, x_0 + K \delta]\). The space \( C(I, J) \) of continuous functions from the interval \( I \) to the interval \( J \) is a complete metric space, by Corollary 2.4. Define \( T : C(I, J) \rightarrow C(I, J) \) by

\[
T(\varphi)(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s) \, ds.
\]

We claim that \( T \) does indeed map \( C(I, J) \) into itself and is a contraction mapping.

Let \( \varphi : I \rightarrow J \) be an element of \( C(I, J) \). Note that if \(|t - t_0| \leq \delta \) then

\[
|(\varphi(t), t) - (x_0, t_0)|^2 = (\varphi(t) - x_0)^2 + (t - t_0)^2 \leq \delta^2 + K^2 \delta^2 < \delta_0^2,
\]

hence \(|F(\varphi(t), t)| \leq K \). It follows from this that

\[
|T(\varphi)(t) - x_0| \leq K \delta
\]

for all \( t \) satisfying \(|t - t_0| < \delta \). The function \( T(\varphi) \) is continuous, and is therefore a well-defined element of \( C(I, J) \) for all \( \varphi \in C(I, J) \).

We now show that \( T \) is a contraction mapping on \( C(I, J) \). Let \( \varphi \) and \( \psi \) be elements of \( C(I, J) \). The hypotheses of the theorem ensure that

\[
|F(\varphi(t), t) - F(\psi(t), t)| \leq M|\varphi(t) - \psi(t)| \leq M \rho(\varphi, \psi)
\]

for all \( t \in I \), where \( \rho(\varphi, \psi) = \sup_{t \in I} |\varphi(t) - \psi(t)| \). Therefore

\[
|T(\varphi)(t) - T(\psi)(t)| = \left| \int_{t_0}^{t} (F(\varphi(s), s) - F(\psi(s), s)) \, ds \right| \leq M|t - t_0| \rho(\varphi, \psi)
\]

for all \( t \) satisfying \(|t - t_0| \leq \delta \). Therefore \( \rho(T(\varphi), T(\psi)) \leq M \delta \rho(\varphi, \psi) \) for all \( \varphi, \psi \in C(I, J) \). But \( \delta \) has been chosen such that \( M \delta < 1 \). This shows that
$T : C(I, J) \to C(I, J)$ is a contraction mapping on $C(I, J)$. It follows from the Contraction Mapping Theorem (Theorem 2.5) that there exists a unique element $\varphi$ of $C(I, J)$ satisfying $T(\varphi) = \varphi$. This function $\varphi$ is the required solution to the differential equation.

A straightforward but somewhat technical least upper bound argument can be used to show that if $x = \psi(t)$ is any other continuous solution to the differential equation
\[
\frac{dx}{dt} = F(x, t)
\]
on the interval $[t_0 - \delta, t_0 + \delta]$ satisfying the initial condition $\psi(t_0) = x_0$, then $|\psi(t) - x_0| \leq K\delta$ for all $t$ satisfying $|t - t_0| \leq \delta$. Thus such a solution to the differential equation must belong to the space $C(I, J)$ defined in the proof of Theorem 2.6. The uniqueness of the fixed point of the contraction mapping $T : C(I, J) \to C(I, J)$ then shows that $\psi = \varphi$, where $\varphi : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ is the solution to the differential equation whose existence was proved in Theorem 2.6. This shows that the solution to the differential equation is in fact unique on the interval $[t_0 - \delta, t_0 + \delta]$. 

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