Course 212: Academic Year 1990-1991
Section 2: Cauchy Sequences and Completeness

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Contents

2 Cauchy Sequences and Completeness

2.1 Cauchy Sequences

Definition Let X be a metric space with distance function d. A sequence $x_1, x_2, x_3, x_4, \ldots$ of points of X is said to be a *Cauchy sequence* in X if and only if, given any $\varepsilon > 0$ there exists some natural number N such that $d(x_j, x_k) < \varepsilon$ for all natural numbers j and k satisfying $j \geq N$ and $k \geq N$.

Every convergent sequence in a metric space is a Cauchy sequence. Indeed let X be a metric space with distance function d, and let $x_1, x_2, x_3, x_4, \ldots$ be a sequence of points in X which converges to some element l of X. Given any $\varepsilon > 0$, there exists some natural number N such that $d(x_n, l) < \varepsilon/2$ whenever $n \geq N$. But then it follows from the Triangle Inequality (Axiom (iii)) that

$$d(x_j, x_k) \leq d(x_j, l) + d(l, x_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $j \geq N$ and $k \geq N$.

Definition A metric space X is said to be *complete* if every Cauchy sequence in X converges to some point of X.

Lemma 2.1 *Let X be a complete metric space, and let A be a closed subset of X. Then A is a complete metric space.*

Proof Let $a_1, a_2, a_3, a_4, \ldots$ be a Cauchy sequence in A. This Cauchy sequence must converge to some point l of X, since X is complete. But then l must be an element of A, by Lemma 1.11, since A is closed in X. We deduce that A is complete, as required. ∎

Example The spaces $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces with respect to the distance function given by $d(z, w) = |z - w|$. Indeed this result is *Cauchy's Criterion for Convergence* (proved in Course 121). However the space $\mathbb{Q}$ of rational numbers (with distance function $d(q, r) = |q - r|$) is not complete. Indeed one can construct an infinite sequence $q_1, q_2, q_3, q_4, \ldots$ of rational numbers which converges (in $\mathbb{R}$) to $\sqrt{2}$. Such a sequence of rational numbers is a Cauchy sequence in both $\mathbb{R}$ and $\mathbb{Q}$. However this Cauchy sequence does not converge to an element of the metric space $\mathbb{Q}$ (since $\sqrt{2}$ is an irrational number). Thus the metric space $\mathbb{Q}$ is not complete.

Lemma 2.2 *The metric space $\mathbb{R}^n$ (with the Euclidean distance function) is a complete metric space.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. We must show that this sequence converges to some element $\mathbf{u}$ of $\mathbb{R}^n$. Let us write $\mathbf{x}_j = (\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{nj})$ for all natural numbers j. Now for any value of m between 1 and n the sequence $\alpha_{m1}, \alpha_{m2}, \alpha_{m3}, \alpha_{m4}, \ldots$ is a Cauchy sequence of real numbers. (This follows easily $|\alpha_{mj} - \alpha_{mk}| \leq |\mathbf{x}_j - \mathbf{x}_k|$.) It follows from Cauchy's Criterion for Convergence that the sequence $\alpha_{m1}, \alpha_{m2}, \alpha_{m3}, \alpha_{m4}, \ldots$ converges to some real number u_m.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. We claim that $\mathbf{x}_j \to \mathbf{u}$ as $j \to +\infty$. Indeed let $\varepsilon > 0$ be given. Then there exist natural numbers $N_1, N_2, \ldots, N_n$ such that $|\alpha_{mj} - u_m| < \varepsilon/\sqrt{n}$ whenever $j \geq N_m$ (where $m = 1, 2, \ldots, n$). Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{u}|^2 = \sum_{m=1}^{n} (\alpha_{mj} - u_m)^2 < \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{u}$ as $j \to +\infty$, as required. ∎

Example The n-sphere S^n (with the chordal distance function given by $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$) is a complete metric space. Indeed S^n is the closed subset of $\mathbb{R}^{n+1}$ given by $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$, and a closed subset of a complete metric space is itself complete (Lemma 2.1).

Let X and Y be metric spaces, with distance functions d_X and d_Y respectively. We say that a function $f\colon X \to Y$ from X to Y is *bounded* if and only if there exists some non-negative constant K such that $d_Y(f(x), f(x')) \leq K$ for all points x and x' in X.

It is easy to see that a function $f\colon X \to \mathbb{R}^n$ is bounded if and only if there exists some non-negative constant K such that $|f(x)| \leq K$ for all $x \in X$.

Let $f\colon X \to Y$ and $g\colon X \to Y$ be bounded functions from X to Y. Then the function $x \mapsto d_Y(f(x), g(x))$ is bounded above on X. Indeed suppose that we choose some point x_0 of X. There exist constants K_1 and K_2 such that $d_Y(f(x), f(x_0)) \leq K_1$ and $d_Y(g(x), g(x_0)) \leq K_2$ for all $x \in X$, since the functions f and g are both bounded. But then

$$\begin{aligned} d_Y(f(x), g(x)) &\leq d_Y(f(x), f(x_0)) + d_Y(f(x_0), g(x_0)) + d_Y(g(x_0), g(x)) \\ &\leq d_Y(f(x_0), g(x_0)) + K_1 + K_2 \end{aligned}$$

for all points x of X, by a straightforward application of the triangle inequality. We define

$$\rho(f, g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

for all bounded functions f and g from X to Y. If f, g, and h are bounded functions from X to Y then

$$d_Y(f(x), h(x)) \leq d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \leq \rho(f, g) + \rho(g, h)$$

for all $x \in X$, and hence $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$, showing that the distance function ρ satisfies the Triangle Inequality.

Theorem 2.3 *Let X and Y be metric spaces with distance functions d_X and d_Y respectively. Suppose that Y is complete. Then the set $C(X, Y)$ of bounded continuous functions from X to Y is a complete metric space with respect to the distance function ρ, where $\rho(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$ for all $f, g \in C(X, Y)$.*

Proof One can readily verify that the metric space axioms are satisfied by the distance function ρ on $C(X, Y)$.

Let $f_1, f_2, f_3, f_4, \ldots$ be a Cauchy sequence in $C(X, Y)$. Choose any $x \in X$. Now $d_Y(f_j(x), f_k(x)) \leq \rho(f_j, f_k)$ for all j and k. Using this inequality, one can easily deduce that $f_1(x), f_2(x), f_3(x), f_4(x), \ldots$ is a Cauchy sequence in Y. But every Cauchy sequence in Y is convergent, since Y is complete. Thus there exists some point $f(x)$ of Y such that $f_n(x) \to f(x)$ as $n \to +\infty$. In this way we define a function $f: X \to Y$ sending any point x of X to the point $f(x)$ defined as above.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $\rho(f_j, f_k) < \varepsilon/3$ whenever $j \geq N$ and $k \geq N$, since the sequence $f_1, f_2, f_3, f_4, \ldots$ is a Cauchy sequence in $C(X, Y)$. Let x be any point of X. Then $d_Y(f_j(x), f_k(x)) < \varepsilon/3$ whenever $j \geq N$ and $k \geq N$. But $d_Y(f_j(x), f_k(x))$ converges to $d_Y(f_j(x), f(x))$ as $k \to +\infty$, by Lemma 1.2. We deduce that, given any $\varepsilon > 0$, there exists some natural number N such that $d_Y(f_j(x), f(x)) \leq \varepsilon/3$ for all $x \in X$ and j satisfying $j \geq N$. In particular

$$\begin{aligned} d_Y(f(x), f(x')) &\leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(x')) + d_Y(f_N(x'), f(x')) \\ &\leq d_Y(f_N(x), f_N(x')) + \frac{2\varepsilon}{3}. \end{aligned}$$

Now the function f_N is bounded. Therefore there exists some non-negative constant K such that $d_Y(f_N(x), f_N(x')) \leq K$ for all $x, x' \in X$. But then

$$d_Y(f(x), f(x')) \leq K + 2\varepsilon/3$$

for all $x, x' \in X$, showing that f is bounded. Also f_N is continuous at x, and hence there exists some $\delta > 0$ such that $d_Y(f_N(x), f_N(x')) < \varepsilon/3$ whenever

$d_X(x, x') < \delta$. It follows that $d_Y(f(x), f(x')) < \varepsilon$ whenever $d_X(x, x') < \delta$. Thus the function f is continuous. We deduce that $f \in C(X, Y)$.

Now we have shown that, given any $\varepsilon > 0$, there exists a natural number N such that $d_Y(f_j(x), f(x)) \leq \varepsilon/3 < \varepsilon$ for all $x \in X$ and j satisfying $j \geq N$. We deduce that $\rho(f_j, f) < \varepsilon$ whenever $j \geq N$. Thus the Cauchy sequence $f_1, f_2, f_3, f_4, \ldots$ converges in $C(X, Y)$ to f, showing that $C(X, Y)$ is complete. ∎

Let F be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then F is a complete metric space, by Lemma 2.1 and Lemma 2.2. The following result therefore follows immediately from Theorem 2.3.

Corollary 2.4 *Let X be a metric space and let F be a closed subset of $\mathbb{R}^n$. Then the space $C(X, F)$ of bounded continuous functions from X to F is a complete metric space with respect to the distance function ρ, where $\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|$ for all $f, g \in C(X, \mathbb{R}^n)$.*

2.2 The Contraction Mapping Theorem

Definition Let X be a metric space with distance function d, and let $T: X \to X$ be a function from X to itself. The function T is said to be a *contraction mapping* if there exists some constant λ satisfying $0 \leq \lambda < 1$ such that $d(T(x), T(x')) \leq \lambda d(x, x')$ for all $x, x' \in X$.

One can readily check that any contraction map $T: X \to X$ on a metric space (X, d) is continuous. Indeed let x be a point of X, and let $\varepsilon > 0$ be given. Then $d(T(x), T(x')) < \varepsilon$ for all points x' of X satisfying $d(x, x') < \varepsilon$.

Theorem 2.5 (Contraction Mapping Theorem) *Let X be a complete metric space, and let $T: X \to X$ be a contraction mapping defined on X. Then T has a unique fixed point in X (i.e., there exists a unique point x of X for which $T(x) = x$).*

Proof Let λ be chosen such that $0 \leq \lambda < 1$ and $d(T(u), T(u')) \leq \lambda d(u, u')$ for all $u, u' \in X$, where d is the distance function on X. First we show the existence of the fixed point x. Let x_0 be any point of X, and define a sequence $x_0, x_1, x_2, x_3, x_4, \ldots$ of points of X by the condition that $x_n = T(x_{n-1})$ for all natural numbers n. It follows by induction on n that $d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$. Using the Triangle Inequality, we deduce that if j and k are natural numbers satisfying $k > j$ then

$$d(x_k, x_j) \leq \sum_{n=j}^{k-1} d(x_{n+1}, x_n) \leq \frac{\lambda^j - \lambda^k}{1 - \lambda} d(x_1, x_0) \leq \frac{\lambda^j}{1 - \lambda} d(x_1, x_0).$$

(Here we have used the identity

$$\lambda^j + \lambda^{j+1} + \cdots + \lambda^{k-1} = \frac{\lambda^j - \lambda^k}{1-\lambda}.)$$

Using the fact that $0 \leq \lambda < 1$, we deduce that the sequence (x_n) is a Cauchy sequence in X. This Cauchy sequence must converge to some point x of X, since X is complete. But then, using Lemma 1.4, we see that

$$T(x) = T\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = x,$$

so that x is a fixed point of T.

If x' were another fixed point of T then we would have

$$d(x', x) = d(T(x'), T(x)) \leq \lambda d(x', x).$$

But this is impossible unless $x' = x$, since $\lambda < 1$. Thus the fixed point x of the contraction map T is unique. ■

2.3 Picard's Theorem

We now use a number of the results of this section in order to prove an existence theorem for solutions of ordinary differential equations known as *Picard's Theorem.*

Theorem 2.6 (Picard's Theorem) *Let $F: U \to \mathbb{R}$ be a continuous function defined over some open set U in the plane $\mathbb{R}^2$, and let (x_0, t_0) be an element of U. Suppose that there exists some non-negative constant M such that*

$$|F(u,t) - F(v,t)| \leq M|u - v| \text{ for all } (u,t) \in U \text{ and } (v,t) \in U.$$

Then there exists a continuous function $\varphi: [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ defined on the interval $[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$ such that $x = \varphi(t)$ is a solution to the differential equation

$$\frac{dx(t)}{dt} = F(x(t), t)$$

with initial condition $x(t_0) = x_0$.

Proof Solving the differential equation with the initial condition $x(t_0) = x_0$ is equivalent to finding a continuous function $\varphi: I \to \mathbb{R}$ satisfying the integral equation

$$\varphi(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

where I denotes the closed interval $[t_0 - \delta, t_0 + \delta]$. (Note that any continuous function φ satisfying this integral equation is automatically differentiable, since the indefinite integral of a continuous function is always differentiable.)

Let $K = |F(x_0, t_0)| + 1$. Using the continuity of the function F, together with the fact that U is open in $\mathbb{R}^2$, one can find some $\delta_0 > 0$ such that the open disk of radius δ_0 about (x_0, t_0) is contained in U and $|F(x, t)| \leq K$ for all points (x, t) in this open disk. Now choose $\delta > 0$ such that

$$\delta\sqrt{1 + K^2} < \delta_0 \text{ and } M\delta < 1.$$

Note that if $|t - t_0| \leq \delta$ and $|x - x_0| \leq K\delta$ then (x, t) belongs to the open disk of radius δ_0 about (x_0, t_0), and hence $(x, t) \in U$ and $|F(x, t)| \leq K$.

Let J denote the closed interval $[x_0 - K\delta, x_0 + K\delta]$. The space $C(I, J)$ of continuous functions from the interval I to the interval J is a complete metric space, by Corollary 2.4. Define $T\colon C(I, J) \to C(I, J)$ by

$$T(\varphi)(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

We claim that T does indeed map $C(I, J)$ into itself and is a contraction mapping.

Let $\varphi\colon I \to J$ be an element of $C(I, J)$. Note that if $|t - t_0| \leq \delta$ then

$$|(\varphi(t), t) - (x_0, t_0)|^2 = (\varphi(t) - x_0)^2 + (t - t_0)^2 \leq \delta^2 + K^2\delta^2 < \delta_0^2,$$

hence $|F(\varphi(t), t)| \leq K$. It follows from this that

$$|T(\varphi)(t) - x_0| \leq K\delta$$

for all t satisfying $|t - t_0| < \delta$. The function $T(\varphi)$ is continuous, and is therefore a well-defined element of $C(I, J)$ for all $\varphi \in C(I, J)$.

We now show that T is a contraction mapping on $C(I, J)$. Let φ and ψ be elements of $C(I, J)$. The hypotheses of the theorem ensure that

$$|F(\varphi(t), t) - F(\psi(t), t)| \leq M|\varphi(t) - \psi(t)| \leq M\rho(\varphi, \psi)$$

for all $t \in I$, where $\rho(\varphi, \psi) = \sup_{t \in I} |\varphi(t) - \psi(t)|$. Therefore

$$\begin{aligned} |T(\varphi)(t) - T(\psi)(t)| &= \left| \int_{t_0}^{t} (F(\varphi(s), s) - F(\psi(s), s))\, ds \right| \\ &\leq M|t - t_0|\rho(\varphi, \psi) \end{aligned}$$

for all t satisfying $|t - t_0| \leq \delta$. Therefore $\rho(T(\varphi), T(\psi)) \leq M\delta\rho(\varphi, \psi)$ for all $\varphi, \psi \in C(I, J)$. But δ has been chosen such that $M\delta < 1$. This shows that

$T: C(I, J) \to C(I, J)$ is a contraction mapping on $C(I, J)$. It follows from the Contraction Mapping Theorem (Theorem 2.5) that there exists a unique element φ of $C(I, J)$ satisfying $T(\varphi) = \varphi$ This function φ is the required solution to the differential equation. ∎

A straightforward but somewhat technical least upper bound argument can be used to show that if $x = \psi(t)$ is any other continuous solution to the differential equation

$$\frac{dx}{dt} = F(x, t)$$

on the interval $[t_0 - \delta, t_0 + \delta]$ satisfying the initial condition $\psi(t_0) = x_0$, then $|\psi(t) - x_0| \leq K\delta$ for all t satisfying $|t - t_0| \leq \delta$. Thus such a solution to the differential equation must belong to the space $C(I, J)$ defined in the proof of Theorem 2.6. The uniqueness of the fixed point of the contraction mapping $T: C(I, J) \to C(I, J)$ then shows that $\psi = \varphi$, where $\varphi: [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ is the solution to the differential equation whose existence was proved in Theorem 2.6. This shows that the solution to the differential equation is in fact unique on the interval $[t_0 - \delta, t_0 + \delta]$.