Course 212: Academic Year 1990-1991
Section 1: Metric Spaces

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Contents

1 Metric Spaces

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X, where this distance function satisfies the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality*. The elements of a metric space are usually referred to as *points* of that metric space.

Note that if X is a metric space with distance function d and if A is a subset of X then the restriction $d|A \times A$ of d to pairs of points of A defines a distance function on A satisfying the axioms for a metric space.

The set $\mathbb{R}$ of real numbers becomes a metric space with distance function d given by $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Similarly the set $\mathbb{C}$ of complex numbers becomes a metric space with distance function d given by $d(z, w) = |z - w|$ for all $z, w \in \mathbb{C}$. Any subset of $\mathbb{R}$ or $\mathbb{C}$ may be regarded as a metric space whose distance function is again given by $d(z, w) = |z - w|$. Further examples of metric spaces are provided by the *Euclidean Spaces* $\mathbb{R}^n$ with the Euclidean distance function.

1.1 Euclidean Spaces

We denote by $\mathbb{R}^n$ the space consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. We can add and subtract elements of $\mathbb{R}^n$ and multiply them by scalars in the usual manner. Thus if $\mathbf{x}$ and $\mathbf{y}$ are elements of $\mathbb{R}^n$ given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n)$$

we define

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n), \qquad \mathbf{x}-\mathbf{y} = (x_1-y_1, x_2-y_2, \ldots, x_n-y_n),$$

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n).$$

The *scalar product* $\mathbf{x}.\mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x}.\mathbf{y} \equiv x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

and the *Euclidean norm* $|\mathbf{x}|$ of $\mathbf{x}$ by

$$|\mathbf{x}| = \sqrt{\mathbf{x}.\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Lemma 1.1 (Schwarz' Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x}.\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

Proof We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}.\mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}.\mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x}.\mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x}.\mathbf{y})^2 + (\mathbf{x}.\mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}.\mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}.\mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x}.\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

It follows easily from Schwarz' Inequality that $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. For

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}).(\mathbf{x} + \mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}.\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

We regard $\mathbb{R}^n$ as a metric space, where the (Euclidean) distance $d(\mathbf{x}, \mathbf{y})$ between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is given by $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. Using the inequalities proved above, one can easily check that all of the metric space axioms are satisfied by this distance function.

If X is any subset of $\mathbb{R}^n$ then the restriction of the Euclidean distance function defined above to pairs of points taken from X defines a distance function on X. In this manner we can regard any subset of $\mathbb{R}^n$ as a metric space in its own right (with respect to the Euclidean distance function on X.

Example The *n-sphere* S^n is defined to be the subset of $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ consisting of all elements $\mathbf{x}$ of $\mathbb{R}^{n+1}$ for which $|\mathbf{x}| = 1$. Thus

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

(Note that S^2 is the standard (2-dimensional) unit sphere in 3-dimensional Euclidean space.) The *chordal distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of S^n is defined to be the length $|\mathbf{x} - \mathbf{y}|$ of the line segment joining $\mathbf{x}$ and $\mathbf{y}$. The n-sphere S^n is a metric space with respect to the chordal distance function.

Example Let $C([a, b])$ denote the set of all continuous real-valued functions on the closed interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Then $C([a, b])$ is a metric space with respect to the distance function ρ, where $\rho(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)|$ for all continuous functions f and g from X to $\mathbb{R}$. (Note that, for all $f, g \in C([a, b])$, $f - g$ is a continuous function on $[a, b]$ and is therefore bounded on $[a, b]$, by a theorem proved in Course 121. Therefore the distance function ρ is well-defined.)

1.2 Convergence and Continuity in Metric Spaces

Definition Let X be a metric space with distance function d. A sequence $x_1, x_2, x_3, x_4, \ldots$ of points in X is said to *converge* to a point l in X if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some natural number N such that $d(x_n, l) < \varepsilon$ whenever $n \geq N$.

We refer to l as the *limit* $\lim\limits_{n \to +\infty} x_n$ of the sequence $x_1, x_2, x_3, x_4, \ldots$.

Note that this definition of convergence generalizes to arbitrary metric spaces the standard definition of convergence for sequences of real or complex numbers.

If a sequence of points in a metric space is convergent then the limit of that sequence is unique. Indeed let $x_1, x_2, x_3, x_4, \ldots$ be a sequence of points in a metric space (X, d) which converges to points l and l' of X. We show that $l = l'$. Now, given any $\varepsilon > 0$, there exist natural numbers N_1 and N_2 such that $d(x_n, l) < \varepsilon$ whenever $n \geq N_1$ and $d(x_n, l') < \varepsilon$ whenever $n \geq N_2$. On choosing n so that $n \geq N_1$ and $n \geq N_2$ we see that

$$0 \leq d(l, l') \leq d(l, x_n) + d(x_n, l') < 2\varepsilon$$

by a straightforward application of the metric space axioms (i)–(iii). Thus $0 \leq d(l, l') < 2\varepsilon$ for every $\varepsilon > 0$, and hence $d(l, l') = 0$, so that $l = l'$ by Axiom (iv).

Lemma 1.2 *Let (X, d) be a metric space, and let $x_1, x_2, x_3, x_4, \ldots$ be a sequence of points of X which converges to some point l of X. Then, for any point y of X, $d(x_n, y) \to d(l, y)$ as $n \to +\infty$.*

Proof Let $\varepsilon > 0$ be given. We must show that there exists some natural number N such that $|d(x_n, y) - d(l, y)| < \varepsilon$ whenever $n \geq N$. However N can be chosen such that $d(x_n, l) < \varepsilon$ whenever $n \geq N$. But

$$d(x_n, y) \leq d(x_n, l) + d(l, y), \qquad d(l, y) \leq d(l, x_n) + d(x_n, y)$$

for all n, hence

$$-d(x_n, l) \leq d(x_n, y) - d(l, y) \leq d(x_n, l)$$

for all n, and hence $|d(x_n, y) - d(l, y)| < \varepsilon$ whenever $n \geq N$, as required. ∎

Definition Let X and Y be metric spaces with distance functions d_X and d_Y respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point x of X if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x' of X satisfying $d_X(x, x') < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at x for every point x of X.

Note that this definition of continuity for functions between metric spaces generalizes the definition of continuity for functions of a real or complex variable.

Lemma 1.3 *Let X, Y and Z be metric spaces with distance functions d_X, d_Y and d_Z respectively, and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then the composition function $g \circ f: X \to Z$ is continuous.*

Proof Let x be any point of X. We show that $g \circ f$ is continuous at x. Let $\varepsilon > 0$ be given. Now the function g is continuous at $f(x)$. Hence there exists some $\eta > 0$ such that $d_Z(g(y), g(f(x))) < \varepsilon$ for all points y of Y satisfying $d_Y(y, f(x)) < \eta$. But then there exists some $\delta > 0$ such that $d_Y(f(x'), f(x)) < \eta$ for all points x' of X satisfying $d_X(x', x) < \delta$. Thus $d_Z(g(f(x')), g(f(x))) < \varepsilon$ for all points x' of X satisfying $d_X(x', x) < \delta$. Thus the composition function $g \circ f$ is continuous at x, as required. ∎

Lemma 1.4 *Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f: X \to Y$ be a continuous function from X to Y. Let $x_1, x_2, x_3, x_4, \ldots$ be a sequence of points of X which converges to some point l of X. Then the sequence $f(x_1), f(x_2), f(x_3), f(x_4), \ldots$ converges to $f(l)$.*

Proof Let $\varepsilon > 0$ be given. We must show that there exists some natural number N such that $d_Y(f(x_n), f(l)) < \varepsilon$ whenever $n \geq N$. However there exists some $\delta > 0$ such that $d_Y(f(x'), f(l)) < \varepsilon$ for all points x' of X satisfying $d_X(x', l) < \delta$, since the function f is continuous at l. Also there exists some natural number N such that $d_X(x_n, l) < \delta$ whenever $n \geq N$, since the sequence $x_1, x_2, x_3, x_4, \ldots$ converges to l. Thus if $n \geq N$ then $d_Y(f(x_n), f(l)) < \varepsilon$, as required. ∎

1.3 Open Sets in Metric Spaces

Definition Let (X, d) be a metric space. Given a point x of X and a non-negative real number r, the *open ball* $B_X(x, r)$ of *radius* r about x in X is defined to be the subset of X given by

$$B_X(x, r) \equiv \{x' \in X : d(x', x) < r\}.$$

One thinks of the open ball $B_X(x, r)$ as representing all points of the metric space X whose distance from x is strictly less than the radius r of the ball.

Definition Let (X, d) be a metric space, and let x be a point of X. A subset N of X is said to be a *neighbourhood* of x (in X) if and only if there exists some $\delta > 0$ such that $B_X(x, \delta) \subset N$, where $B_X(x, \delta)$ is the open ball of radius δ about x in X.

Definition Let (X, d) be a metric space. A subset V of X is said to be an *open set* if and only if the following condition is satisfied:

> given any point v of V there exists some $\delta > 0$ such that $B_X(v, \delta) \subset V$.

Thus a subset V of X is an open set if and only if V is a neighbourhood of v for all points v of V. By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in the case when V is the empty set.)

Lemma 1.5 *Let X be a metric space with distance function d, and let x_0 be a point of X. Then, for any positive real number r, the open ball $B_X(x_0, r)$ of radius r about x_0 is an open set in X.*

Proof Let x be an element of $B_X(x_0, r)$. We must show that there exists some $\delta > 0$ such that $B_X(x, \delta) \subset B_X(x_0, r)$. Now $d(x, x_0) < r$, and hece $\delta > 0$, where $\delta = r - d(x, x_0)$. Moreover if $x' \in B_X(x, \delta)$ then

$$d(x', x_0) \leq d(x', x) + d(x, x_0) < \delta + d(x, x_0) = r,$$

by the Triangle Inequality, and thus $x' \in B_X(x_0, r)$. We deduce that $B_X(x, \delta) \subset B_X(x_0, r)$. This shows that $B_X(x_0, r)$ is an open set, as required. ∎

Lemma 1.6 *Let X be a metric space with distance function d, and let x_0 be a point of X. Then, for any non-negative real number r, the set $\{x \in X : d(x, x_0) > r\}$ is an open set in X.*

Proof Let x be a point of X satisfying $d(x, x_0) > r$, and let y be any point of X satisfying $d(y, x) < \delta$, where $\delta = d(x, x_0) - r$. Then

$$d(x, x_0) \leq d(x, y) + d(y, x_0),$$

by the Triangle Inequality, and therefore

$$d(y, x_0) \geq d(x, x_0) - d(x, y) > d(x, x_0) - \delta = r.$$

Thus $B_X(x, \delta) \subset \{x' \in X : d(x', x_0) > r\}$, showing that $\{x \in X : d(x, x_0) > r\}$ is an open set in X. ∎

Proposition 1.7 *Let X be a metric space. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open sets;

(ii) the union of any collection of open sets is itself an open set;

(iii) the intersection of any finite *collection of open sets is itself an open set.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. Thus (i) is satisfied.

Let $\mathcal{A}$ be any collection of open sets in the metric space X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself an open set. Let x be an element of U. Then x belongs to V for some open set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that the open ball $B_X(x, \delta)$ is a subset of V. But $V \subset U$, and thus $B_X(x, \delta) \subset U$. This shows that U is an open set in the metric space X. Thus (ii) is satisfied.

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in the metric space X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these open sets. Let x be an element of V. Now x belongs to V_j for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of open sets.) Moreover $B_X(x, \delta) \subset B_X(x, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(x, \delta) \subset V$. This shows that the intersection V of the open sets $V_1, V_2, \ldots, V_k$ is itself an open set in the metric space X. Thus (iii) is satisfied. ∎

Remark For each natural number n, let V_n denote the open set in the complex plane $\mathbb{C}$ defined by

$$V_n = \{z \in \mathbb{C} : |z| < 1/n\}.$$

The intersection of all of these sets (as n ranges over the set of natural numbers) consists of the set $\{0\}$, and this set is not an open subset of the complex plane. This demonstrates that an intersection of an infinite number of open sets in a metric space is not necessarily an open set.

Lemma 1.8 *Let X be a metric space. A sequence $(x_j : j \in \mathbb{N})$ of points in X converges to a point l if and only if, given any open set U which contains l, there exists some natural number N such that the point x_j belongs to U for all j satisfying $j \geq N$.*

Proof Let $(x_j : j \in \mathbb{N})$ be a sequence with the property that, given any open set U which contains l, there exists some natural number N such that the point x_j belongs to U whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B_X(l, \varepsilon)$ of radius ε about l is an open set by Lemma 1.5. Therefore there exists some natural number N such that x_j belongs to $B_X(l, \varepsilon)$ whenever $j \geq N$. Thus $d(x_j, l) < \varepsilon$ whenever $j \geq N$. This shows that the sequence (x_j) converges to l.

Conversely, suppose that the sequence (x_j) converges to l. Let U be an open set which contains l. Then there exists some $\varepsilon > 0$ such that the open ball $B_X(l, \varepsilon)$ of radius ε about l is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points x of X that satisfy $d(x, l) < \varepsilon$. But there exists some natural number N with the property that $d(x_j, l) < \varepsilon$ whenever $j \geq N$, since the sequence (x_j) converges to l. Therefore x_j belongs to U for all j satisfying $j \geq N$, as required. ∎

1.4 Closed Sets in a Metric Space

A subset F of a metric space X is said to be a *closed set* in X if and only if its complement $X \setminus F$ is open. (Recall that the *complement* $X \setminus F$ of F in X is, by definition, the set of all points of the metric space X that do not belong to F.) The following result follows immediately from Lemma 1.5, Lemma 1.6, and the definition of closed sets.

Lemma 1.9 *Let X be a metric space with distance function d, and let x_0 be a point of X. Given any non-negative real number r, the sets*

$$\{x \in X : d(x, x_0) \leq r\}, \qquad \{x \in X : d(x, x_0) \geq r\}$$

are closed. In particular, the set $\{x_0\}$ consisting of the single point x_0 is a closed set in X.

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets). The following result therefore follows directly from the definition of closed sets and Proposition 1.7.

Proposition 1.10 *Let X be a metric space. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed sets;

(ii) the intersection of any collection of closed sets in X is itself a closed set;

(iii) the union of any finite *collection of closed sets in X is itself a closed set.*

Lemma 1.11 *Let F be a closed set in a metric space X and let $(x_j : j \in \mathbb{N})$ be a sequence of points of F. Suppose that $x_j \to l$ as $j \to +\infty$. Then l also belongs to F.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that l were a point belonging to $X \setminus F$. It would then follow from Lemma 1.8 that $x_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $x_j \in F$ for all j. This contradiction shows that l must belong to F, as required. ∎

1.5 Continuous Functions and Open and Closed Sets

Let X and Y be metric spaces, and let $f: X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point x of X if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x'), f(x)) < \varepsilon$ for all points x' of X satisfying $d_X(x', x) < \delta$, where d_X and d_Y denote the distance functions on X and Y respectively. Expressed in terms of open balls, this means that the function $f: X \to Y$ is continuous at x if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(x, \delta)$ into $B_Y(f(x), \varepsilon)$ (where $B_X(x, \delta)$ and $B_Y(f(x), \varepsilon)$ denote the open balls of radius δ and ε about x and $f(x)$ respectively).

Let $f: X \to Y$ be a function from a set X to a set Y. Given any subset V of Y, we denote by $f^{-1}(V)$ the *preimage* of V under the map f, defined by

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

Proposition 1.12 *Let X and Y be metric spaces, and let $f: X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is an open set in X for every open set V of Y.*

Proof Suppose that $f: X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let x be a point belonging to $f^{-1}(V)$. We must show that there exists some $\delta > 0$ with the property that $B_X(x, \delta) \subset f^{-1}(V)$. Now $f(x)$ belongs to V. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(x), \varepsilon) \subset V$. But f is continuous at x. Therefore there exists some $\delta > 0$ such that f maps the open ball $B_X(x, \delta)$ into $B_Y(f(x), \varepsilon)$ (see the remarks above). Thus $f(x') \in V$ for all $x' \in B_X(x, \delta)$, showing that $B_X(x, \delta) \subset f^{-1}(V)$. We have thus shown that if $f: X \to Y$ is continuous then $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f: X \to Y$ has the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let x be any point of X. We must show that f is continuous at x. Let $\varepsilon > 0$ be given. The open ball $B_X(f(x), \varepsilon)$ is an open set in Y, by Lemma 1.5, hence $f^{-1}(B_Y(f(x), \varepsilon))$ is an open set in X which contains x. It follows that there exists some $\delta > 0$ such that $B_X(x, \delta) \subset f^{-1}(B_Y(f(x), \varepsilon))$. We have thus shown that, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps the open ball $B_X(x, \delta)$ into $B_Y(f(x), \varepsilon)$. We conclude that f is continuous at x, as required. ∎

Let $f: X \to Y$ be a function between metric spaces X and Y. Then the preimage $f^{-1}(Y \setminus G)$ of the complement $Y \setminus G$ of any subset G of Y is equal to the complement $X \setminus f^{-1}(G)$ of the preimage $f^{-1}(G)$ of G. Indeed

$$x \in f^{-1}(Y \setminus G) \iff f(x) \in Y \setminus G \iff f(x) \notin G \iff x \notin f^{-1}(G).$$

Also a subset of a metric space is closed if and only if its complement is open. The following result therefore follows directly from Propostion 1.12.

Corollary 1.13 *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(G)$ is a closed set in X for every closed set G in Y.*

Let $f\colon X \to Y$ be a continuous function from a metric space X to a metric space Y. Then, for any point y of Y, the set $\{x \in X : f(x) = y\}$ is a closed subset of X. This follows from Corollary 1.13, together with the fact that the set $\{y\}$ consisting of the single point y is a closed subset of the metric space Y.

Let X be a metric space, and let $f\colon X \to \mathbb{R}$ be a continuous function from X to $\mathbb{R}$. Then, given any real number c, the sets

$$\{x \in X : f(x) > c\}, \qquad \{x \in X : f(x) < c\}$$

are open subsets of X, and the sets

$$\{x \in X : f(x) \geq c\}, \qquad \{x \in X : f(x) \leq c\}, \qquad \{x \in X : f(x) = c\}$$

are closed subsets of X. Also, given real numbers a and b satisfying $a < b$, the set

$$\{x \in X : a < f(x) < b\}$$

is an open subset of X, and the set

$$\{x \in X : a \leq f(x) \leq b\}$$

is a closed subset of X.

Similar results hold for continuous functions $f\colon X \to \mathbb{C}$ from X to $\mathbb{C}$. Thus, for example,

$$\{x \in X : |f(x)| < R\}, \qquad \{x \in X : |f(x)| > R\}$$

are open subsets of X and

$$\{x \in X : |f(x)| \leq R\}, \qquad \{x \in X : |f(x)| \geq R\}, \qquad \{x \in X : |f(x)| = R\}$$

are closed subsets of X, for any non-negative real number R.

1.6 Homeomorphisms

Let X and Y be metric spaces. A function $h: X \to Y$ from X to Y is said to be a *homeomorphism* if it is a bijection and both $h: X \to Y$ and its inverse $h^{-1}: Y \to X$ are continuous. If there exists a homeomorphism $h: X \to Y$ from a metric space X to a metric space Y, then the metric spaces X and Y are said to be *homeomorphic*.

Example The interval $(-1, 1)$ and the real line $\mathbb{R}$ are homeomorphic. A homeomorphism $h: (-1, 1) \to \mathbb{R}$ is given by $h(t) = t/(1 - t^2)$ for all $t \in (-1, 1)$. The inverse $h^{-1}: \mathbb{R} \to (-1, 1)$ of h is the continuous function given by

$$h^{-1}(s) = \begin{cases} \dfrac{-1 + \sqrt{1 + 4s^2}}{2s} & \text{if } s \neq 0; \\ 0 & \text{if } s = 0. \end{cases}$$

The following result follows directly on applying Proposition 1.12 to $h: X \to Y$ and to $h^{-1}: Y \to X$.

Lemma 1.14 *Let X and Y be metric spaces, and let $h: X \to Y$ be a homeomorphism. Then the homomorphism h induces a one-to-one correspondence between that open sets of X and the open sets of Y: a subset V of Y is open in Y if and only if $h^{-1}(V)$ is open in X.*

Lemma 1.15 *Let X and Y be metric spaces, and let $h: X \to Y$ be a homeomorphism. Let Z be a metric space. A function $f: Y \to Z$ is continuous if and only if $f \circ h: X \to Z$ is continuous.*

Proof If $f: Y \to Z$ is continuous then $f \circ h: X \to Z$ is continuous, since a composition of continuous functions is continuous, by Lemma 1.3. Conversely if $f \circ h: X \to Z$ is continuous then $f: Y \to Z$ is continuous, since $f = (f \circ h) \circ h^{-1}$. ∎

Lemma 1.16 *Let X and Y be metric spaces, and let $h: X \to Y$ be a homeomorphism. A sequence $x_1, x_2, x_3, x_4, \ldots$ of points in X is convergent in X if and only if the corresponding sequence $h(x_1), h(x_2), h(x_3), h(x_4), \ldots$ is convergent in Y.*

Proof This result follows from a direct application of Lemma 1.4 to $h: X \to Y$ and its inverse $h^{-1}: Y \to X$. ∎

1.7 Continuity of Functions into Euclidean Spaces

Let $f: X \to \mathbb{R}^n$ be a function mapping a metric space X into $\mathbb{R}^n$. The components $f_1, f_2, \ldots, f_n$ of f are the real-valued functions on X defined such that

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$.

Proposition 1.17 *A function $f: X \to \mathbb{R}^n$ from a metric space X to $\mathbb{R}^n$ is continuous if and only if its components $f_1, f_2, \ldots, f_n$ are continuous functions from X to $\mathbb{R}$.*

Proof Note that $f_i = p_i \circ f$ for $i = 1, 2, \ldots, n$, where $p_i: \mathbb{R}^n \to \mathbb{R}$ is the function which projects a point $(y_1, y_2, \ldots, y_n)$ on $\mathbb{R}^n$ onto its ith coordinate y_i. Now p_i is continuous for all i. Also any composition of continuous functions is continuous, by Lemma 1.3. Thus if f is continuous then its components $f_1, f_2, \ldots, f_n$ are all continuous.

Conversely suppose that $f_1, f_2, \ldots, f_n$ are all continuous at some point x_0 of X. Let $\varepsilon > 0$ be given. Then there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(x) - f_i(x_0)| < \varepsilon/\sqrt{n}$ for all points x of X satisfying $d_X(x, x_0) < \delta_i$, where d_X denotes the distance function on X. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If x is any point of X satisfying $d_X(x, x_0) < \delta$ then

$$|f(x) - f(x_0)|^2 = \sum_{i=1}^{n} |f_i(x) - f_i(x_0)|^2 < \varepsilon^2,$$

and thus $|f(x) - f(x_0)| < \varepsilon$. This shows that the function f is continuous at x_0, as required. ∎

Lemma 1.18 *The functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $p(x, y) = xy$ are continuous.*

Proof First we show that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at every point (u, v) of $\mathbb{R}^2$. Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$ and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that the function s is continuous at (u, v).

Next we show that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at every point (u, v) of $\mathbb{R}^2$. Let $\varepsilon > 0$ be given. Let

$$\delta = \min\left(\frac{\varepsilon}{2(|v| + 1)}, \frac{\varepsilon}{2(|u| + 1)}, 1\right).$$

If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and thus

$$|x - u| < \frac{\varepsilon}{2(|v| + 1)}, \qquad |y - v| < \frac{\varepsilon}{2(|u| + 1)}, \qquad |y| < |v| + 1.$$

But then

$$|p(x, y) - p(u, v)| = |xy - uv| = |(x - u)y + u(y - v)| \leq |x - u||y| + |u||y - v| < \varepsilon.$$

This shows that the function p is continuous at (u, v). ∎

Proposition 1.19 *Let X be a metric space, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f.g$ are continuous. If in addition $g(x) \neq 0$ for all $x \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f.g = p \circ h$, where $h: X \to \mathbb{R}^2$, $s: \mathbb{R}^2 \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ are given by $h(x) = (f(x), g(x))$, $s(u, v) = u + v$ and $p(u, v) = uv$ for all $x \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 1.17, Lemma 1.18 Lemma 1.3 that $f + g$ and $f.g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continous.

Now suppose that $g(x) \neq 0$ for all $x \in X$. Note that $1/g = r \circ g$, where $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Let B^n denote the open unit ball in $\mathbb{R}^n$, defined by

$$B^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\},$$

and let $h: B^n \to \mathbb{R}^n$ be the function defined by

$$h(\mathbf{x}) = \frac{1}{1 - |\mathbf{x}|^2}\mathbf{x}.$$

The function h is a bijection from B^n to $\mathbb{R}^n$ whose inverse $h^{-1}: \mathbb{R}^n \to B$ is given by

$$h^{-1}(\mathbf{x}) = \begin{cases} \dfrac{-1 + \sqrt{1 + 4|\mathbf{x}|^2}}{2|\mathbf{x}|^2}\mathbf{x} & \text{if } \mathbf{x} \neq 0; \\ \mathbf{0} & \text{if } \mathbf{x} = 0. \end{cases}$$

Note that $h^{-1}(\mathbf{x}) = \varphi(|\mathbf{x}|)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is the continuous function defined by

$$\varphi(t) = \begin{cases} \dfrac{-1 + \sqrt{1 + 4t^2}}{2t^2} & \text{if } t \neq 0; \\ 1 & \text{if } t = 0. \end{cases}$$

(The continuity of φ at 0 follows from an application of l'Hôpital's Rule.) A straighforward application of Lemma 1.3, Proposition 1.17 and Proposition 1.19 shows that h and h^{-1} are continuous. Thus $h\colon B^n \to \mathbb{R}^n$ is a homeomorphism from B^n to $\mathbb{R}^n$.

Example Let S^n denote the n-sphere, defined by

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\},$$

and let N be the point of S^n with coordinates $(0, 0, \ldots, 0, 1)$. We show that $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$. Define a function $h\colon S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$h(x_1, x_2, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right).$$

The function h is a bijection whose inverse $h^{-1}\colon \mathbb{R}^n \to S^n \setminus \{N\}$ is given by

$$h^{-1}(y_1, y_2, \ldots, y_n) = \left(\frac{2y_1}{|\mathbf{y}|^2 + 1}, \frac{2y_2}{|\mathbf{y}|^2 + 1}, \ldots, \frac{2y_n}{|\mathbf{y}|^2 + 1}, \frac{|\mathbf{y}|^2 - 1}{|\mathbf{y}|^2 + 1} \right)$$

(where $|\mathbf{y}|^2 = y_1^2 + y_2^2 + \cdots + y_n^2$). A straightforward application of Propositions 1.17 and 1.19 shows that the functions h and h^{-1} are continuous. We deduce that $h\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is a homeomorphism from $S^n \setminus \{N\}$ to $\mathbb{R}^n$. This construction described in this example is referred to as *stereographic projection* of $S^n \setminus \{N\}$ onto $\mathbb{R}^n$.