Course 212, 1989–90, Scholarship Examination (SF)

1. Consider the following subsets of $\mathbb{R}^3$. Determine which are open and which are closed in $\mathbb{R}^3$.

   (i) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 7\}$,

   (ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \geq 7 \text{ and } z \leq 0\}$,

   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \geq 7 \text{ or } z > 0\}$,

   (iv) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } y^2 + z^2 = 1/x\}$,

2. (a) Define what is meant by saying that a metric space (X, d) is *complete*.

   (b) State and prove the *Contraction Mapping Theorem* (also known as the *Banach Contraction Principle*).

3. (a) Define the concept of a *topological space*.

   (b) Let (X, d) be a metric space. A subset V of X is said to be *open* in X if and only if, given any $v \in V$, there exists some $\delta > 0$ such that $\{x \in X : d(x, v) < \delta\} \subset V$. Prove that X, with these open sets, is a topological space.

   (c) Let X be a topological space. Prove that a subset V of X is open if and only if, given any $v \in V$, there exists some open set N such that $v \in N$ and $N \subset V$.

4. Let X be a topological space, and let A be a subset of X.

   (a) Give the definition of the *closure* $\overline{A}$ of A.

   (b) Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of points of A. Suppose that $\lim_{n \to +\infty} a_n = l$ for some $l \in X$. Prove that l belongs to the closure $\overline{A}$ of A.

   (c) What is the closure in $\mathbb{R}$ of the set $\{1/n : n \in \mathbb{N}\}$?

   (d) Let x be a point of X. Prove that $x \in \overline{A}$ if and only if $U \cap A$ is non-empty for all open sets U in X containing x.

5. (a) Define the *product topology* on the Cartesian Product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces $X_1, X_2, \ldots, X_n$.

(b) Prove that the topology on n-dimensional Euclidean space $\mathbb{R}^n$ induced by the Euclidean distance function coincides with the product topology obtained on regarding $\mathbb{R}^n$ as the Cartesian product of n copies of $\mathbb{R}$.

6. (a) Let X be a topological space. Define what is meant by saying that X is *compact*.

   (b) Let $f\colon X \to Y$ be a continuous function, and let A be a compact subset of X. Prove that the image $f(A)$ of A under the map f is compact.

   (c) Let K be a compact subset of a Hausdorff space X. Prove that K is closed.

   (d) Let X be a compact topological space, let Y be a Hausdorff space, and let $f\colon X \to Y$ be a continuous function. Prove that $f(A)$ is closed in Y for every closed set A in X.

7. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be continuous functions from $\mathbb{R}^n$ into $\mathbb{R}$. Suppose that $f(\mathbf{x}) > 0$ and $g(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Suppose also that $f(\lambda\mathbf{x}) = \lambda f(\mathbf{x})$ and $g(\lambda\mathbf{x}) = \lambda^3 g(\mathbf{x})$ for all $\lambda > 0$. Prove that there exist constants c and C satisfying $0 < c \leq C$ such that $c|\mathbf{x}|^2 f(\mathbf{x}) \leq g(\mathbf{x}) \leq C|\mathbf{x}|^2 f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

8. (a) Let X be a topological space. Define what is meant by saying that X is *connected*. Prove that X is connected if and only if every continuous function $f\colon X \to \mathbb{Z}$ is constant.

   (b) A topological space X is said to be *path-connected* if, given any two points x_0 and x_1 of X, there exists a continuous function $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Prove that every path-connected topological space is connected.

   (c) Let X be connected topological space. Suppose that every point of X has an open neighbourhood homeomorphic to an open set in $\mathbb{R}^n$. Prove that X is path-connected. [Hint: choose $x_0 \in X$ and show that the sets

$$\begin{aligned} U_0 &= \{x \in X : x \text{ can be joined to } x_0 \text{ by a path}\} \\ U_0 &= \{x \in X : x \text{ cannot be joined to } x_0 \text{ by a path}\} \end{aligned}$$

   are both open in X.]

9. (a) Let M be a metric space. Define what is meant by saying that M is a *topological manifold* of dimension n.

(b) Define the concept of a *continuous coordinate system* $(x^1, x^2, \ldots, x^n)$ on a topological manifold M of dimension n. Define what is meant by saying that two continuous coordinate systems $(x^1, x^2, \ldots, x^n)$ and $(y^1, y^2, \ldots, y^n)$ are *smoothly compatible*.

(c) Let $\mathcal{A}$ be a collection of continuous coordinate systems on a topological manifold M of dimension n. Define what is meant by saying that $\mathcal{A}$ is a *smooth atlas* on M.

(d) Let $\mathcal{A}$ be a smooth atlas on a topological manifold M of dimension n. Prove that $\mathcal{A}$ is contained in an atlas $\mathcal{A}_{\max}$ which is *maximal* in the following sense: if $(x^1, x^2, \ldots, x^n)$ is any continuous coordinate system that is smoothly compatible with all the coordinate systems belonging to $\mathcal{A}_{\max}$ then $(x^1, x^2, \ldots, x^n)$ must itself belong to $\mathcal{A}_{\max}$.

(e) Define the concept of a *smooth manifold* of dimension n.

10. Let X be a topological space, and let x_0 be a point of X. Define the *fundamental group* $\pi_1(X, x_0)$ of X based at x_0, and prove that $\pi_1(X, x_0)$ is indeed a well-defined group.