1. Consider the following subsets of $\mathbb{R}^3$. Determine which are open and which are closed in $\mathbb{R}^3$.
   (i) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 7\}$,
   (ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \geq 7 \text{ and } z \leq 0\}$,
   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \geq 7 \text{ or } z > 0\}$,
   (iv) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } y^2 + z^2 = 1/x\}$.

2. (a) Define what is meant by saying that a metric space $(X, d)$ is complete.
   (b) State and prove the Contraction Mapping Theorem (also known as the Banach Contraction Principle).

3. (a) Define the concept of a topological space.
   (b) Let $(X, d)$ be a metric space. A subset $V$ of $X$ is said to be open in $X$ if and only if, given any $v \in V$, there exists some $\delta > 0$ such that $\{x \in X : d(x, v) < \delta\} \subset V$. Prove that $X$, with these open sets, is a topological space.
   (c) Let $X$ be a topological space. Prove that a subset $V$ of $X$ is open if and only if, given any $v \in V$, there exists some open set $N$ such that $v \in N$ and $N \subset V$.

4. Let $X$ be a topological space, and let $A$ be a subset of $X$.
   (a) Give the definition of the closure $\overline{A}$ of $A$.
   (b) Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of points of $A$. Suppose that $\lim_{n \to +\infty} a_n = l$ for some $l \in X$. Prove that $l$ belongs to the closure $\overline{A}$ of $A$.
   (c) What is the closure in $\mathbb{R}$ of the set $\{1/n : n \in \mathbb{N}\}$?
   (d) Let $x$ be a point of $X$. Prove that $x \in \overline{A}$ if and only if $U \cap A$ is non-empty for all open sets $U$ in $X$ containing $x$.

5. (a) Define the product topology on the Cartesian Product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces $X_1, X_2, \ldots, X_n$. 

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(b) Prove that the topology on \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) induced by the Euclidean distance function coincides with the product topology obtained on regarding \( \mathbb{R}^n \) as the Cartesian product of \( n \) copies of \( \mathbb{R} \).

6. (a) Let \( X \) be a topological space. Define what is meant by saying that \( X \) is compact.

(b) Let \( f: X \to Y \) be a continuous function, and let \( A \) be a compact subset of \( X \). Prove that the image \( f(A) \) of \( A \) under the map \( f \) is compact.

(c) Let \( K \) be a compact subset of a Hausdorff space \( X \). Prove that \( K \) is closed.

(d) Let \( X \) be a compact topological space, let \( Y \) be a Hausdorff space, and let \( f: X \to Y \) be a continuous function. Prove that \( f(A) \) is closed in \( Y \) for every closed set \( A \) in \( X \).

7. Let \( f: \mathbb{R}^n \to \mathbb{R} \) and \( g: \mathbb{R}^n \to \mathbb{R} \) be continuous functions from \( \mathbb{R}^n \) into \( \mathbb{R} \). Suppose that \( f(x) > 0 \) and \( g(x) > 0 \) for all \( x \in \mathbb{R}^n \setminus \{0\} \). Suppose also that \( f(\lambda x) = \lambda f(x) \) and \( g(\lambda x) = \lambda^3 g(x) \) for all \( \lambda > 0 \). Prove that there exist constants \( c \) and \( C \) satisfying \( 0 < c \leq C \) such that \( c|x|^2 f(x) \leq g(x) \leq C|x|^2 f(x) \) for all \( x \in \mathbb{R}^n \).

8. (a) Let \( X \) be a topological space. Define what is meant by saying that \( X \) is connected. Prove that \( X \) is connected if and only if every continuous function \( f: X \to \mathbb{Z} \) is constant.

(b) A topological space \( X \) is said to be path-connected if, given any two points \( x_0 \) and \( x_1 \) of \( X \), there exists a continuous function \( \gamma: [0, 1] \to X \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). Prove that every path-connected topological space is connected.

(c) Let \( X \) be connected topological space. Suppose that every point of \( X \) has an open neighbourhood homeomorphic to an open set in \( \mathbb{R}^n \). Prove that \( X \) is path-connected. [Hint: choose \( x_0 \in X \) and show that the sets

\[
U_0 = \{ x \in X : x \text{ can be joined to } x_0 \text{ by a path} \} \\
U_0 = \{ x \in X : x \text{ cannot be joined to } x_0 \text{ by a path} \}
\]

are both open in \( X \).]

9. (a) Let \( M \) be a metric space. Define what is meant by saying that \( M \) is a topological manifold of dimension \( n \).
(b) Define the concept of a continuous coordinate system \((x^1, x^2, \ldots, x^n)\) on a topological manifold \(M\) of dimension \(n\). Define what is meant by saying that two continuous coordinate systems \((x^1, x^2, \ldots, x^n)\) and \((y^1, y^2, \ldots, y^n)\) are smoothly compatible.

(c) Let \(\mathcal{A}\) be a collection of continuous coordinate systems on a topological manifold \(M\) of dimension \(n\). Define what is meant by saying that \(\mathcal{A}\) is a smooth atlas on \(M\).

(d) Let \(\mathcal{A}\) be a smooth atlas on a topological manifold \(M\) of dimension \(n\). Prove that \(\mathcal{A}\) is contained in an atlas \(\mathcal{A}_{\text{max}}\) which is maximal in the following sense: if \((x^1, x^2, \ldots, x^n)\) is any continuous coordinate system that is smoothly compatible with all the coordinate systems belonging to \(\mathcal{A}_{\text{max}}\) then \((x^1, x^2, \ldots, x^n)\) must itself belong to \(\mathcal{A}_{\text{max}}\).

(e) Define the concept of a smooth manifold of dimension \(n\).

10. Let \(X\) be a topological space, and let \(x_0\) be a point of \(X\). Define the fundamental group \(\pi_1(X, x_0)\) of \(X\) based at \(x_0\), and prove that \(\pi_1(X, x_0)\) is indeed a well-defined group.