Course 212: Academic Year 1989-1990
Section 9: Winding Numbers

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Contents

9 Winding Numbers

9.1 Winding Numbers of Closed Curves in the Plane

Let $\gamma: [0,1] \to \mathbb{C}$ be a continuous closed curve in the complex plane which is defined on some closed interval $[0,1]$ (so that $\gamma(0) = \gamma(1)$), and let w be a complex number which does not belong to the image of the closed curve γ. The map $p_w: \mathbb{C} \to \mathbb{C} \setminus \{w\}$ defined by $p_w(z) = w + \exp(2\pi i z)$ is a covering map. Observe that if z_1 and z_2 are complex numbers then $p_w(z_1) = p_w(z_2)$ if and only if $z_1 - z_2$ is an integer. Using the Path Lifting Property for covering maps (Theorem 8.3) we see that there exists a continuous path $\tilde{\gamma}: [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $p_w \circ \tilde{\gamma} = \gamma$. Let us define

$$n(\gamma, w) = \tilde{\gamma}(1) - \tilde{\gamma}(0).$$

Now $p_w(\tilde{\gamma}(1)) = p_w(\tilde{\gamma}(0))$ (since $\gamma(1) = \gamma(0)$). It follows from this that $n(\gamma, w)$ is an integer. We claim that the value of $n(\gamma, w)$ is independent of the choice of the path $\tilde{\gamma}$ on $\mathbb{C}$.

Let $\sigma: [0,1] \to \mathbb{C}$ be a continuous path in $\mathbb{C}$ with the property that $p_w \circ \sigma = \gamma$. Then $p_w(\sigma(t)) = p_w(\tilde{\gamma}(t))$ for all $t \in [0,1]$, and hence $\sigma(t) - \tilde{\gamma}(t)$ is an integer for all $t \in [0,1]$. But the map sending $t \in [0,1]$ to $\sigma(t) - \tilde{\gamma}(t)$ is continuous on $[0,1]$; therefore this map must be a constant map. Thus there exists some integer m with the property that $\sigma(t) = \tilde{\gamma}(t) + m$ for all $t \in [0,1]$, and hence

$$\sigma(1) - \sigma(0) = \tilde{\gamma}(1) - \tilde{\gamma}(0).$$

This proves that the value of $n(\gamma, w)$ is independent of the choice of the lift $\tilde{\gamma}$ of the closed curve γ.

Definition Let $\gamma: [0,1] \to \mathbb{C}$ be a continuous closed curve in the complex plane, and let w be a complex number which does not belong to the image of the closed curve γ. Then the *winding number* $n(\gamma, w)$ of the closed curve γ about w is defined by

$$n(\gamma, w) \equiv \tilde{\gamma}(1) - \tilde{\gamma}(0),$$

where $\tilde{\gamma}: [0,1] \to \mathbb{C}$ is some continuous path in $\mathbb{C}$ with the property that

$$\gamma(t) = w + \exp(2\pi i \tilde{\gamma}(t))$$

for all $t \in [0,1]$.

Theorem 9.1 *Let w be a complex number and let $\gamma_0: [0,1] \to \mathbb{C}$ and $\gamma_1: [0,1] \to \mathbb{C}$ be closed curves in $\mathbb{C}$ which do not pass through w. Suppose that there exists some homotopy $F: [0,1] \times [0,1] \to \mathbb{C}$ with the following properties:*

(i) $F(t, 0) = \gamma_0(t)$ *and* $F(t, 1) = \gamma_1(t)$ *for all* $t \in [0, 1]$*,*

(ii) $F(0, \tau) = F(1, \tau)$ *for all* $\tau \in [0, 1]$*,*

(iii) the complex number w *does not belong to the image* $F([0, 1] \times [0, 1])$ *of the homotopy* F*.*

Then $n(\gamma_0, w) = n(\gamma_1, w)$ *(where* $n(\gamma_0, w)$ *and* $n(\gamma_1, w)$ *are the winding numbers of the closed curves* γ_0 *and* γ_1*).*

Proof It follows from the Path Lifting Property for covering maps (Theorem 8.3) that there exists a continuous map $g: [0, 1] \to \mathbb{C}$ such that $p_w(g(t)) = F(t, 0)$ for all $t \in [0, 1]$ (where $p_w(z) = w + \exp(2\pi i z)$ for all $z \in \mathbb{C}$). It then follows from the Homotopy Lifting Property (Theorem 8.4) that there exists a continuous map $\tilde{F}: [0, 1] \times [0, 1] \to \mathbb{C}$ such that $\tilde{F}(t, 0) = g(t)$ and $p_w \circ \tilde{F} = F$. Now the path $t \mapsto \tilde{F}(t, \tau)$ is a lift of the loop γ_τ, where $\gamma_\tau: [0, 1] \to \mathbb{C}$ is the loop given by $\gamma_\tau(t) = F(t, \tau)$. We deduce that $n(\gamma_\tau, w) = \tilde{F}(1, \tau) - \tilde{(}0, \tau)$ for all $\tau \in [0, 1]$. This implies that the function $\tau \mapsto n(\gamma_\tau, w)$ is a continuous function on the interval $[0, 1]$ taking values in the set $\mathbb{Z}$ of integers. But such a function must be constant on $[0, 1]$. Thus $n(\gamma_0, w) = n(\gamma_1, w)$, as required. ∎

Corollary 9.2 *Let* $\gamma_0: [0, 1] \to \mathbb{C}$ *and* $\gamma_1: [0, 1] \to \mathbb{C}$ *be continuous closed curves in* $\mathbb{C}$*, and let* w *be a complex number which does not lie on the images of the closed curves* γ_0 *and* γ_1*. Suppose that, for all* $t \in [0, 1]$*, the line segment in the complex plane* $\mathbb{C}$ *joining* $\gamma_0(t)$ *to* $\gamma_1(t)$ *does not pass through* w*. Then* $n(\gamma_0, w) = n(\gamma_1, w)$*.*

Proof Let $F: [0, 1] \times [0, 1] \to \mathbb{C}$ be the homotopy defined by

$$F(t, \tau) = (1 - \tau)\gamma_0(t) + \tau\gamma_1(t)$$

for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Note that w does not lie on the image of the homotopy F. We can therefore apply Theorem 9.1 to conclude that $n(\gamma_0, w) = n(\gamma_1, w)$. ∎

Corollary 9.3 (Dog-Walking Principle) *Let* $\gamma_0: [0, 1] \to \mathbb{C}$ *and* $\gamma_1: [0, 1] \to \mathbb{C}$ *be continuous closed curves in* $\mathbb{C}$*, and let* w *be a complex number which does not lie on the images of the closed curves* γ_0 *and* γ_1*. Suppose that* $|\gamma_1(t) - \gamma_0(t)| < |\gamma_0(t) - w|$ *for all* $t \in [0, 1]$*. Then* $n(\gamma_0, w) = n(\gamma_1, w)$*.*

Proof The inequality $|\gamma_1(t) - \gamma_0(t)| < |\gamma_0(t) - w|$ ensures that the line segment in $\mathbb{C}$ joining $\gamma_0(t)$ and $\gamma_1(t)$ does not pass through w. The result therefore follows directly from Corollary 9.2. ∎

Corollary 9.4 *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$, and let $\sigma\colon [0,1] \to \mathbb{C}$ be a continuous path in $\mathbb{C}$ whose image does not intersect the image of γ. Then $n(\gamma, \sigma(0)) = n(\gamma, \sigma(1))$. Thus the function $w \mapsto n(\gamma, w)$ is constant over each path-component of the set $\mathbb{C} \setminus \gamma([0,1])$.*

Proof Let $F\colon [0,1] \times [0,1] \to \mathbb{C}$ be defined by $F(t,\tau) = \gamma(t) - \sigma(\tau)$ for all $(t,\tau) \in [0,1] \times [0,1]$. Then $F(t,\tau) \neq 0$ for all $(t,\tau) \in [0,1] \times [0,1]$, since the path σ does not intersect the closed curve γ. It follows from Theorem 9.1 that $n(\gamma_0, 0) = n(\gamma_1, 0)$, where

$$\gamma_\tau(t) = F(t,\tau) = \gamma(t) - \sigma(\tau)$$

for all $t \in [0,1]$. Thus

$$n(\gamma, \sigma(0)) = n(\gamma_0, 0) = n(\gamma_1, 0) = n(\gamma, \sigma(1)). \quad \blacksquare$$

9.2 The Fundamental Theorem of Algebra

Theorem 9.5 (The Fundamental Theorem of Algebra) *Let $P\colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial with complex coefficients. Then there exists some complex number z_0 such that $P(z_0) = 0$.*

Proof The result is trivial if $P(0) = 0$. Thus it suffices to prove the result when $P(0) \neq 0$.

For any $r > 0$, let the closed curve σ_r denote the circle about zero of radius r, traversed once in the anticlockwise direction, given by $\sigma_r(t) = r\exp(2\pi i t)$ for all $t \in [0,1]$. Consider the winding number $n(P \circ \sigma_r, 0)$ of $P \circ \sigma_r$ about zero. We claim that this winding number is equal to m for large values of r, where m is the degree of the polynomial P.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_n$ are complex numbers, and where $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and

$$Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}.$$

Let R be defined by $R = |a_m|^{-1}(|a_0| + |a_1| + \cdots + |a_m|)$ If $|z| > R$ then $|z| > 1$ and hence

$$\left|\frac{Q(z)}{P_m(z)}\right| = \frac{1}{|a_m z|}\left|\frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1}\right| < 1.$$

Thus if $|z| > R$ then $|P(z) - P_m(z)| < |P_m(z)|$. It follows from the Dog-Walking Principle (Corollary 9.3) that $n(P \circ \sigma_r, 0) = n(P_m \circ \sigma_r, 0) = m$ for all r satisfying $r > R$.

Now were it the case that the polynomial P were everywhere non-zero then the closed curves $P \circ \sigma_r$ would be homotopic in $\mathbb{C} \setminus \{0\}$ for all $r \geq 0$ and thus $n(P \circ \sigma_r, 0)$ would be a constant function of r (Theorem 9.1). Now $P \circ \sigma_0$ is the constant curve at $P(0)$, and hence $n(P \circ \sigma_0, 0) = 0$. Thus if P were everywhere non-zero then $n(P \circ \sigma_r, 0) = 0$ for all $r \geq 0$. But $n(P \circ \sigma_r, 0) = m$ if $r > R$, where m is the degree of the non-constant polynomial P. Since $m > 0$, we conclude that P has at least one zero in the complex plane. ∎

9.3 The Kronecker Principle

The proof of the Fundamental Theorem of Algebra given above depends on continuity of the polynomial P, together with the fact that $n(P \circ \sigma_r, 0)$ is non-zero for sufficiently large r, where σ_r denotes the circle of radius r about zero, described once in thef anticlockwise direction. We can therefore generalize the proof of the Fundamental Theorem of Algebra in order to obtain the following result, known as the *Kronecker Principle*.

Theorem 9.6 (Kronecker Principle) *Let $f: D \to \mathbb{C}$ be a continuous map defined on the closed unit disk D in $\mathbb{C}$. Let w be a complex number which does not lie on the image $f(\partial D)$ of the boundary circle ∂D of D. Suppose that $n(f \circ \sigma, w) \neq 0$, where $\sigma: [0,1] \to \partial D$ is the parameterization of the boundary circle ∂D of D defined by $\sigma(t) = \exp(2\pi i t)$. Then there exists some $z \in D \setminus \partial D$ with the property that $f(z) = w$.*

Proof Suppose that it were the case that $f(z) \neq w$ for all $z \in D$. It would then follow from Theorem 9.1 that $n(f \circ \sigma_r, w)$ would be a constant function of r for all $r \in [0,1]$, where $\sigma_r: [0,1] \to D$ is given by $\sigma_r(t) = r\exp(2\pi i t)$. But $n(f \circ \sigma_0, w) = 0$, since $(f \circ \sigma_0)(t) = f(0)$ for all $t \in [0,1]$, and $n(f \circ \sigma_1, w) \neq 0$ by assumption. This contradiction shows that $f(z) = w$ for some $z \in D$. ∎

9.4 The Borsuk-Ulam Theorem

Lemma 9.7 *Let $f: S^1 \to \mathbb{C}$ be a continuous function defined on S^1, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. Then the winding number $n(f \circ \sigma, 0)$ of $f \circ \sigma$ about 0 is odd, where $\sigma: [0,1] \to S^1$ is given by $\sigma(t) = \exp(2\pi i t)$.*

Proof Let $\tilde{\gamma}: [0,1] \to \mathbb{C}$ be a continuous path in $\mathbb{C}$ such that $\exp(2\pi i\tilde{\gamma}(t)) = f(\sigma(t))$ for all $t \in [0,1]$. (The existence of $\tilde{\gamma}$ is guaranteed by the Path Lifting Property, applied to the covering map from $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$ sending $z \in \mathbb{C}$ to $\exp(2\pi i z)$.) Now $f(\sigma(t + \frac{1}{2})) = -f(\sigma(t))$ for all $t \in [0, \frac{1}{2}]$, since

$\sigma(t+\frac{1}{2}) = -\sigma(t)$ and $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. Thus $\exp(2\pi i\tilde{\gamma}(t+\frac{1}{2})) = \exp(2\pi i(\tilde{\gamma}(t)+\frac{1}{2}))$ for all $t \in [0, \frac{1}{2}]$. It follows that $\tilde{\gamma}(t+1\ over 2) = \tilde{\gamma}(t)+m+\frac{1}{2}$ for some integer m. (The value of m for which this identity is valid does not depend on t, since every continuous function from $[0, \frac{1}{2}]$ to the set of integers is necessarily constant.) Hence

$$n(f \circ \sigma, 0) = (\tilde{\gamma}(1) - \tilde{\gamma}(0)) = (\tilde{\gamma}(1) - \tilde{\gamma}(\tfrac{1}{2})) - (\tilde{\gamma}(\tfrac{1}{2}) - \tilde{\gamma}(0)) = 2m + 1.$$

Thus $n(f \circ \sigma, 0)$ is an odd integer, as required. ∎

We shall identify the space $\mathbb{R}^2$ with $\mathbb{C}$, identifying $(x, y) \in \mathbb{R}^2$ with the complex number $x + iy \in \mathbb{C}$ for all $x, y \in \mathbb{R}$. This is permissible, since we are interested in purely topological results concerning continuous functions defined on appropriate subsets of these spaces. Under this identification the closed unit disk D is given by

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

As usual, we define

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Lemma 9.8 *Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. Then there exists some point $\mathbf{n}_0$ of S^2 with the property that $f(\mathbf{n}_0) = 0$.*

Proof Let $\varphi\colon D \to S^2$ be the map defined by

$$\varphi(x, y) = (x, y, +\sqrt{1 - x^2 - y^2}).$$

(Thus the map φ maps the closed disk D homeomorphically onto the upper hemisphere in $\mathbb{R}^3$.) Let $\sigma\colon [0, 1] \to S^2$ be the parameterization of the equator in S^2 defined by

$$\sigma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

for all $t \in [0, 1]$. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. The winding number $n(f \circ \sigma, 0)$ is an odd integer, by Lemma 9.7, and is thus non-zero. It follows from the Kronecker Principle (Theorem 9.6), applied to $f \circ \varphi\colon D \to \mathbb{R}^2$, that there exists some point (u, v) of D such that $f(\varphi(u, v)) = 0$. Thus $f(\mathbf{n}_0) = 0$, where $\mathbf{n}_0 = \sigma(u, v)$. ∎

We conclude immediately from this result that there are no continuous maps $f\colon S^2 \to S^1$ from the 2-sphere S^2 to the circle S^1 with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$.

Theorem 9.9 (Borsuk-Ulam) *Let $f: S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists some point $\mathbf{n}$ of S^2 with the property that $f(-\mathbf{n}) = f(\mathbf{n})$.*

Proof This result follows immediately on applying Lemma 9.8 to the continuous function $g: S^2 \to \mathbb{R}^2$ defined by $g(\mathbf{n}) = f(\mathbf{n}) - f(-\mathbf{n})$. ∎

Remark It is possible to generalize the Borsuk-Ulam Theorem to n dimensions. Let S^n be the unit n-sphere centered on the origin in $\mathbb{R}^n$. The Borsuk-Ulam Theorem in n-dimensions states that if $f: S^n \to \mathbb{R}^n$ is a continuous map then there exists some point $\mathbf{x}$ of S^n with the property that $f(\mathbf{x}) - f(-\mathbf{x})$.

9.5 Winding Numbers and Contour Integrals

A continuous curve is said to be *piecewise C^1* if it is made up of a finite number of continuously differentiable segments. We now show how the winding number of a piecewise C^1 closed curve in the complex plane can be expressed as a contour integral.

Proposition 9.10 *Let $\gamma: [0,1] \to \mathbb{C}$ be a piecewise C^1 closed curve in the complex plane, and let w be a point of $\mathbb{C}$ that does not lie on the curve γ. Then the winding number $n(\gamma, w)$ of γ about w is given by*

$$n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}.$$

Proof By definition $n(\gamma, w) = \alpha(1) - \alpha(0)$, where $\alpha: [0,1] \to \mathbb{C}$ is a path in $\mathbb{C}$ such that

$$\gamma(t) = w + \exp(2\pi i \alpha(t))$$

for all $t \in [0,1]$. Taking derivatives, we see that

$$\gamma'(t) = 2\pi i \exp(2\pi i \alpha(t)) \alpha'(t) = 2\pi i (\gamma(t) - w) \alpha'(t).$$

Thus

$$n(\gamma, w) = \alpha(1) - \alpha(0) = \int_0^1 \alpha'(t)\, dt = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)\, dt}{\gamma(t) - w} = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}.$$ ∎