Course 212: Academic Year 1989-1990
Section 8: Covering Maps

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Contents

8 Covering Maps

8.1 Evenly Covered Open Sets and Covering Maps

Definition Let X and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p.

Definition Let X and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. The map $p: \tilde{X} \to X$ is said to be a *covering map* over the topological space X if and only if the following conditions are satisfied:

(i) the map $p: \tilde{X} \to X$ is surjective,

(ii) every point of X has an open neighbourhood which is evenly covered by the map p.

If $p: \tilde{X} \to X$ is a covering map over a topological space X then the topological space $\tilde{X}$ is said to be a *covering space* of X.

Example Let S^1 be the unit circle in $\mathbb{R}^2$. Then the map $p: \mathbb{R} \to S^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

is a covering map. Indeed let $\mathbf{n}$ be a point of S^1. Consider the open neighbourhood U of $\mathbf{n}$ in S^1 defined by $U = S^1 \setminus \{-\mathbf{n}\}$. Now $\mathbf{n} = (\cos 2\pi t_0, \sin 2\pi t_0)$ for some $t_0 \in \mathbb{R}$. Then $p^{-1}(U)$ is the union of the disjoint open sets J_n for all integers n, where

$$J_n = \{t \in \mathbb{R} : t_0 + n - \tfrac{1}{2} < t < t_0 + n + \tfrac{1}{2}\}.$$

Each of the open sets J_n is mapped homeomorphically onto U by the map p. This shows that $p: \mathbb{R} \to S^1$ is a covering map.

Example The map $p: \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined by $p(z) = \exp(2\pi i z)$ is a covering map. Given any $\theta \in [-\pi, \pi]$ let us define

$$U_\theta = \{z \in \mathbb{C} \setminus \{0\} : \arg z \neq -\theta\}.$$

Note the U_θ is evenly covered by the map p. Indeed $p^{-1}(U_\theta)$ consists of the union of the open sets

$$\{z \in \mathbb{C} : \frac{\theta}{2\pi} + n - \tfrac{1}{2} < \operatorname{Re} z < \frac{\theta}{2\pi} + n + \tfrac{1}{2}\},$$

where each of these open sets is mapped homeomorphically onto U_θ by the map p.

Example Let S^1 denote the unit circle in $\mathbb{R}^2$. Let n be a non-zero integer. Let $\beta_n\colon S^1 \to S^1$ be defined by

$$\beta_n(\cos\theta, \sin\theta) = (\cos n\theta, \sin n\theta).$$

Then $\beta_n\colon S^1 \to S^1$ is a covering map.

Example Let $\mathbb{R}P^n$ denote the *real projective* n*-space*. This is the topological space obtained from the n-sphere S^n by identifying antipodal points on S^n. (We regard S^n as the unit n-sphere in $\mathbb{R}^{n+1}$ consisting of all $\mathbf{x} \in \mathbb{R}^{n-1}$ satisfying $|\mathbf{x}| = 1$. We define an equivalence relation on S^n where distinct points $\mathbf{x}$ and $\mathbf{y}$ of S^n are equivalent if and only if $\mathbf{x} = -\mathbf{y}$. The space $\mathbb{R}P^n$ is then defined to be the set of equivalence classes of points of S^n under this equivalence relation. The topology on $\mathbb{R}P^n$ is the quotient topology induced by the quotient map $\rho\colon S^n \to \mathbb{R}P^n$. Thus a subset U of $\mathbb{R}P^n$ is open if and only if $\rho^{-1}(U)$ is open in S^n.) It is easily verified that the quotient map $\rho\colon S^n \to \mathbb{R}P^n$ is a covering map.

Example Consider the map $\alpha\colon (-2,2) \to S^1$ defined by $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in (-2,2)$. It can easily be shown that the point $(1,0)$ of S^1 has no open neighbourhood which is evenly covered by the map α. Indeed suppose that there existed an open neighbourhood N of $(1,0)$ which was evenly covered by α. Then there would exist some δ satisfying $0 < \delta < \frac{1}{2}$ such that $U_\delta \subset N$, where

$$U_\delta = \{(\cos 2\pi t, \sin 2\pi t) : -\delta < t < \delta\}.$$

The open set U_δ would then be evenly covered by the map α. However the connected components of $\alpha^{-1}(U_\delta)$ are $(-2,-2+\delta)$, $(-1-\delta,-1+\delta)$, $(-\delta,\delta)$, $(1-\delta,1+\delta)$ and $(2-\delta,2)$, and neither $(-2,-2+\delta)$ nor $(2-\delta,2)$ is mapped homeomorphically onto U_δ by α.

8.2 The Path Lifting and Homotopy Lifting Properties

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a topological space, and let $f\colon Z \to X$ be a continuous map from Z to X. A continuous map $\tilde{f}\colon Z \to \tilde{X}$ is said to be a *lift* of the map $f\colon Z \to X$ if and only if $p \circ \tilde{f} = f$. We shall prove various results concerning the existence and uniqueness of such lifts.

Proposition 8.1 *Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a connected topological space, and let $\tilde{f}\colon Z \to \tilde{X}$ and $\tilde{g}\colon Z \to \tilde{X}$ be continuous maps. Suppose that $p \circ \tilde{f} = p \circ \tilde{g}$ and that $\tilde{f}(z_0) = \tilde{g}(z_0)$ for some $z_0 \in Z$. Then $\tilde{f} = \tilde{g}$.*

Proof Let $Z_0 = \{z \in Z : \tilde{f}(z) = \tilde{g}(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed.

Let z be a point of Z. There exists an open neighbourhood U of $p(\tilde{f}(z))$ in X which is evenly covered by the map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the map p. One of these open sets contains $\tilde{f}(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $\tilde{g}(z)$; let this open set be denoted by $\tilde{V}$. Note that $\tilde{U} = \tilde{V}$ if $z \in Z_0$ (so that $\tilde{f}(z) = \tilde{g}(z)$), and $\tilde{U} \cap \tilde{V} = \emptyset$ if $z \in Z \setminus Z_0$ (so that $\tilde{f}(z) \neq \tilde{g}(z)$). Let $N_z = \tilde{f}^{-1}(\tilde{U}) \cap \tilde{g}^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. In this case $\tilde{V} = \tilde{U}$, so that $\tilde{f}(N_z) \subset \tilde{U}$ and $\tilde{g}(N_z) \subset \tilde{U}$. But $p \circ \tilde{f} = p \circ \tilde{g}$, and the restriction $p|\tilde{U}$ of the map p to $\tilde{U}$ maps $\tilde{U}$ homeomorphically onto U. Therefore $\tilde{f}|N_z = \tilde{g}|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$. But $\tilde{f}(N_z) \subset \tilde{U}$ and $\tilde{g}(N_z) \subset \tilde{V}$. Therefore $\tilde{f}(z') \neq \tilde{g}(z')$ for all $z' \in N_z$, so that $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open, so that Z_0 is closed.

The subset Z_0 of Z is both open and closed. Also Z_0 is non-empty, since there exists some point z_0 of Z for which $\tilde{f}(z_0) = \tilde{g}(z_0)$. It follows from the connectedness of Z that $Z_0 = Z$. Therefore $\tilde{f} = \tilde{g}$. ∎

Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a topological space, let A be a subset of Z, and let $f: Z \to X$ and $g: A \to \tilde{X}$ be continuous maps with the property that $p \circ g = f|A$. In this situation one can ask whether or not the map g can be extended to a map $\tilde{f}: Z \to \tilde{X}$ such that $p \circ \tilde{f} = f$. A problem of this sort is referred to as a *lifting problem*; and a map $\tilde{f}$ with the desired properties is referred to as a *lift* of the map f to $\tilde{X}$. The next lemma proves the existence of a lift in the special case when A is connected and $f(Z)$ is contained wholly within some open set in X that is evenly covered by the map p. This result is then used to derive more general lifting theorems.

Lemma 8.2 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a topological space, let A be a connected subset of Z, and let $f: Z \to X$ and $g: A \to \tilde{X}$ be continuous maps with the property that $p \circ g = f|A$. Suppose that $f(Z) \subset U$, where U is an open subset of X that is evenly covered by the map p. Then there exists a continuous map $\tilde{f}: Z \to \tilde{X}$ such that $\tilde{f}|A = g$ and $p \circ \tilde{f} = f$.*

Proof Choose $a_0 \in A$. Now U is evenly covered by the map p. Therefore $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the map p. One of these open sets contains $g(a_0)$; let this set be denoted by $\tilde{U}$. Let $s: U \to \tilde{U}$ be the inverse of the homeomorphism $p|\tilde{U}: \tilde{U} \to U$. Now $p \circ (s \circ f)|A = f|A = p \circ g$, and $s(f(a_0)) = g(a_0)$. It follows from Proposition 8.1 that $s \circ f|A = g$, since A is connected. Define $\tilde{f} = s \circ f$. Then $\tilde{f}|A = g$ and $p \circ \tilde{f} = f$, as required. ∎

Theorem 8.3 (Path Lifting Property) *Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Let $\gamma: [0, 1] \to X$ be a continuous path in X, and let w be a point of $\tilde{X}$ for which $p(w) = \gamma(0)$. Then there exists a unique continuous path $\tilde{\gamma}: [0, 1] \to \tilde{X}$ such that $\tilde{\gamma}(0) = w$ and $p \circ \tilde{\gamma} = \gamma$.*

Proof We can cover X by a collection $\mathcal{U}$ of open sets, each of which is evenly covered by the map p, since $p: \tilde{X} \to X$ is a covering map. Now the collection of sets of the form $\gamma^{-1}(U)$ with $U \in \mathcal{U}$ is an open cover of the interval $[0, 1]$. Now $[0, 1]$ is compact, by the Heine-Borel Theorem (Theorem 4.2). It follows from the Lebesgue Lemma (Lemma 4.20) that there exists some $\delta > 0$ such that every subinterval of length less than δ is mapped by γ into one of the open sets belonging to $\mathcal{U}$. Partition the interval $[0, 1]$ into subintervals $[t_{i-1}, t_i]$, where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, and where the length of each subinterval is less than δ. Now it follows from Lemma 8.2 that once $\tilde{\gamma}(t_{i-1})$ has been determined, we can extend $\tilde{\gamma}$ continuously over the ith subinterval $[t_{i-1}, t_i]$. Thus by extending $\tilde{\gamma}$ successively over $[t_0, t_1]$, $[t_1, t_2]$,…, $[t_{n-1}, t_n]$, we can lift the path $\gamma: [0, 1] \to X$ to a path $\tilde{\gamma}: [0, 1] \to \tilde{X}$ starting at w. The uniqueness of $\tilde{\gamma}$ follows from Proposition 8.1. ∎

Theorem 8.4 (Homotopy Lifting Property) *Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Let $F: [0, 1] \times [0, 1] \to X$ and $g: [0, 1] \to \tilde{X}$ be continuous maps with the property that $p(g(t)) = F(t, 0)$ for all $t \in [0, 1]$. Then g can be extended to a unique continuous map $\tilde{F}: [0, 1] \times [0, 1] \to \tilde{X}$ such that $\tilde{F}(t, 0) = g(t)$ for all $t \in [0, 1]$ and $p \circ \tilde{F} = F$.*

Proof We can cover X by a collection $\mathcal{U}$ of open sets, each of which is evenly covered by the map p. The collection of sets of the form $F^{-1}(U)$ with $U \in \mathcal{U}$ is an open cover of the square $[0, 1] \times [0, 1]$. An application of the Lebesgue Lemma (as in the proof of Theorem 8.3) shows that there exists some $\delta > 0$ with the property that any square contained in $[0, 1] \times [0, 1]$ whose sides have length less than δ is mapped by F into some open set in X which is evenly covered by the map p. It follows from Lemma 8.2 that if the lift $\tilde{F}$ of F has already been determined over a corner, or along one side, or along two adjacent sides of a square whose sides have length less than δ, then $\tilde{F}$ can be extended over the whole of that square. Thus if we subdivide the square $[0,1] \times [0,1]$ into smaller squares whose sides have length less than δ then we can extend the map g to a lift $\tilde{F}$ of F by successively extending $\tilde{F}$ in turn over each of the smaller squares. The uniqueness of $\tilde{F}$ follows from Proposition 8.1. ∎

Theorem 8.4 is in fact a special case of a more general Homotopy Lifting Property. Let $p: \tilde{X} \to X$ be a covering map over a topological space X, let Z be a topological space, and let $F: Z \times [0,1] \to X$ and $g: Z \to \tilde{X}$ be continuous maps with the property that $F(z,0) = p(g(z))$ for all $z \in Z$. The Homotopy Lifting Property states that, under these conditions, there exists a map $\tilde{F}: Z \times [0,1] \to \tilde{X}$ such that $\tilde{F}(z,0) = g(z)$ for all $z \in Z$ and $p \circ \tilde{F} = F$.

8.3 The Fundamental Group of the Circle

Theorem 8.5 *Let $b \in S^1$ be some chosen basepoint of the circle S^1. Then $\pi_1(S^1, b) \cong \mathbb{Z}$.*

Proof We regard S^1 as the unit circle in $\mathbb{R}^2$. Let $p: \mathbb{R} \to S^1$ be the map defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t).$$

The map p is a covering map. We can take $b = (1,0) = p(0)$. We define a function $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ as follows: given a loop $\gamma: [0,1] \to S^1$ based at b we define $\lambda([\gamma]) = \tilde{\gamma}(1)$, where $\tilde{\gamma}: [0,1] \to \mathbb{R}$ is the (unique) lift of γ starting at 0 (so that $\tilde{\gamma}(0) = 0$ and $p \circ \tilde{\gamma} = \gamma$).

First we note that $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is well-defined. If $\tilde{\gamma}$ is the lift of the loop γ starting at 0 then $\tilde{\gamma}(1) \in \mathbb{Z}$, since $p(\tilde{\gamma}(1)) = \gamma(1) = b$. Suppose that γ_0 and γ_1 are loops in S^1 based at b which represent the same element of $\pi_1(S^1, b)$. This means that there exists a homotopy $F: [0,1] \times [0,1] \to S^1$ such that $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$ for all $t \in [0,1]$, and $F(0,\tau) = F(1,\tau) = b$ for all $\tau \in [0,1]$. It follows from the Homotopy Lifting Property (Theorem 8.4) that there exists a lift $\tilde{F}: [0,1] \times [0,1] \to \mathbb{R}$ of F such that $\tilde{F}(0,\tau) = 0$ for all $\tau \in [0,1]$, and $p \circ \tilde{F} = F$. Note in particular that the (unique) lifts $\tilde{\gamma}_0: [0,1] \to \mathbb{R}$ and $\tilde{\gamma}_1: [0,1] \to \mathbb{R}$ of the paths γ_0 and γ_1 respectively starting at 0 are given by $\tilde{\gamma}_0(t) = \tilde{F}(t,0)$ and $\tilde{\gamma}_1(t) = \tilde{F}(t,1)$ for all $t \in [0,1]$. Now the function $\tau \mapsto \tilde{F}(1,\tau)$ takes values in $\mathbb{Z}$, since $p(\tilde{F}(1,\tau)) = F(1,\tau) = b$, and $p^{-1}(\{b\}) = \mathbb{Z}$. But every continuous integer-valued function on $[0,1]$ is constant. Therefore $\tilde{F}(1,\tau)$ is a constant function of τ. Thus

$$\tilde{\gamma}_0(1) = \tilde{F}(1,0) = \tilde{F}(1,1) = \tilde{\gamma}_1(1).$$

Thus the function $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is well-defined.

Next we show that λ is a homomorphism. Let α and β be loops based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be the lifts of α and β respectively starting at 0. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha.\beta$, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Define a continuous path $\sigma: [0,1] \to \mathbb{R}$ by

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \tilde{\beta}(2t-1) + \tilde{\alpha}(1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$). Then $p \circ \sigma = \alpha.\beta$ and $\sigma(0) = 0$. It follows that

$$\lambda([\alpha][\beta]) = \lambda([\alpha.\beta]) = \sigma(1) = \tilde{\alpha}(1) + \tilde{\beta}(1) = \lambda([\alpha]) + \lambda([\beta]).$$

Thus λ is a homomorphism from the group $\pi_1(S^1, b)$ to the additive group $\mathbb{Z}$.

Let α and β be loops in S^1 based at b. Suppose that $\lambda([\alpha]) = \lambda([\beta])$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$, where $\tilde{\alpha}$ and $\tilde{\beta}$ be the lifts of α and β respectively starting at 0. Define

$$F(t, \tau) = p\left((1-\tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right).$$

Then $F: [0,1] \times [0,1] \to S^1$ is a homotopy between α and β, with

$$F(0, \tau) = b = F(1, \tau) \qquad (\tau \in [0,1]).$$

Thus $\alpha \simeq \beta$ rel $\{0,1\}$, so that $[\alpha] = [\beta]$. This shows that $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

Given any $n \in \mathbb{N}$, let $\gamma_n: [0,1] \to S^1$ be the loop based at b defined by

$$\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt).$$

Then $\lambda([\gamma_n]) = n$. This shows that the homomorphism λ is surjective. We conclude that $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism. ∎

Let D denote the closed unit disk in $\mathbb{R}^2$, given by $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. We now show that every continuous map from D to itself has at least one fixed point.

Theorem 8.6 (The Brouwer Fixed Point Theorem) *Let $f: D \to D$ be a continuous map which maps the closed unit disk D into itself. Then there exists some $\mathbf{x}_0 \in D$ such that $f(\mathbf{x}_0) = \mathbf{x}_0$.*

Proof Let S^1 denote the boundary circle of D, and let $i\colon S^1 \hookrightarrow D$ denote the inclusion map from S^1 to D. This inclusion map induces a corresponding homomorphism $i_\#\colon \pi_1(S^1, \mathbf{b}) \to \pi_1(D, \mathbf{b})$ of fundamental groups, where $\mathbf{b} \in S^1$ is some suitably chosen basepoint.

Suppose that it were the case that the map f has no fixed point in D. Then one could define a continuous map $r\colon D \to S^1$ as follows: for each $\mathbf{x} \in D$, let $r(\mathbf{x})$ be the point on the boundary S^1 of D obtained by continuing the line segment joining $f(\mathbf{x})$ to $\mathbf{x}$ beyond $\mathbf{x}$ until it intersects S^1 at the point $r(\mathbf{x})$. Note that $r|S^1$ is the identity map of S^1.

Let $r_\#\colon \pi_1(D, \mathbf{b}) \to \pi_1(S^1, \mathbf{b})$ be the homomorphism of fundamental groups induced by $r\colon D \to S^1$. Now $(r \circ i)_\#\colon \pi_1(S^1, \mathbf{b}) \to \pi_1(S^1, \mathbf{b})$ is the identity isomorphism of $\pi_1(S^1, \mathbf{b})$, since $r \circ i\colon S^1 \to S^1$ is the identity map. But $(r \circ i)_\# = r_\# \circ i_\#$ (see Lemma 7.5). It follows that $i_\#\colon \pi_1(S^1, \mathbf{b}) \to \pi_1(D, \mathbf{b})$ is injective, and $r_\#\colon \pi_1(D, \mathbf{b}) \to \pi_1(S^1, \mathbf{b})$ is surjective. But this is impossible, since $\pi_1(S^1, \mathbf{b}) \cong \mathbb{Z}$ (Theorem 8.5) and $\pi_1(D, \mathbf{b})$ is the trivial group. This contradiction shows that the continuous map $f\colon D \to D$ must have at least one fixed point. ∎

Remark The Brouwer Fixed Point Theorem is also valid in higher dimensions. This theorem states that any continuous map from the closed n-dimensional ball into itself must have at least one fixed point. The proof of the theorem for $n > 2$ is analogous to the proof for $n = 2$, but with the fundamental groups of the ball and its boundary sphere being replaced by another topologicical invariant, namely the n-dimensional *homology groups* of the closed ball and its boundary sphere.

8.4 Covering Maps over Simply-Connected Spaces

Theorem 8.7 *Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Suppose that $\tilde{X}$ is path-connected and that X is simply-connected. Then the covering map $p\colon \tilde{X} \to X$ is a homeomorphism.*

Proof The map $p\colon \tilde{X} \to X$ is surjective (since covering maps are by definition surjective). We must show that it is also injective. Let w_0 and w_1 be points of $\tilde{X}$ for which $p(w_0) = p(w_1)$. There exists a continuous path $\sigma\colon [0,1] \to \tilde{X}$ with $\sigma(0) = w_0$ and $\sigma(1) = w_1$, since $\tilde{X}$ is path-connected. Then $p \circ \sigma$ is a loop in X based at the point x_0, where $x_0 = p(w_0)$. But every loop based at x_0 is homotopic (rel $\{0,1\}$) to the constant loop based at x_0. Thus there exists a continuous homotopy $F\colon [0,1] \times [0,1] \to X$ such that $F(t,0) = p(\sigma(t))$ and $F(t,1) = x_0$ for all $t \in [0,1]$, and $F(0,\tau) = F(1,\tau) = x_0$. Now it follows from the Homotopy Lifting Property (Theorem 8.4) that there exists a lift

$\tilde{F}\colon [0,1] \times [0,1] \to \tilde{X}$ of F such that $\tilde{F}(t,0) = \sigma(t)$ for all $t \in [0,1]$ and $p \circ \tilde{F} = F$. Now it follows from the uniqueness of lifts of paths that the lift to $\tilde{X}$ of a constant path in X is a constant path in $\tilde{X}$. It follows that $\tilde{F}$ is constant on the three sides of the square $[0,1] \times [0,1]$ that are mapped by p onto the point x_0. Thus

$$w_0 = \sigma(0) = F(0,0) = F(0,1) = F(1,1) = F(1,0) = \sigma(1) = w_1.$$

This shows that $p\colon \tilde{X} \to X$ is injective.

Now the covering map $p\colon \tilde{X} \to X$ is a bijection. It is a straightforward exercise to verify that $p(V)$ is open in $\tilde{X}$ for every open set V in X. Thus $p^{-1}\colon X \to \tilde{X}$ is continuous. We conclude that $p\colon \tilde{X} \to X$ is a homeomorphism. ∎