Course 212: Academic Year 1989-1990
Section 6: Smooth Manifolds and Partitions of Unity

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Contents

6 Smooth Manifolds and Partitions of Unity

6.1 Smooth Manifolds

Definition A metric space M is said to be a *topological manifold* of dimension n if and only if every point of M has an open neighbourhood homeomorphic to an open set in $\mathbb{R}^n$.

Let M be a topological manifold. A *continuous coordinate system* defined over an open set U in M is defined to be an n-tuple $(x^1, x^2, \ldots, x^n)$ of continuous real-valued functions on U such that the map $\varphi \colon U \to \mathbb{R}^n$ defined by

$$\varphi(u) = \big(x^1(u), x^2(u), \ldots, x^n(u)\big)$$

maps U homeomorphically onto some open set in $\mathbb{R}^n$. The domain U of the coordinate system $(x^1, x^2, \ldots, x^n)$ is referred to as a *coordinate patch* on M.

Let $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$ be continuous coordinate systems defined over coordinate patches U and V respectively. We say that the coordinate systems $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$ are *smoothly compatible*) if and only if, on the overlap $U \cap V$ of the coordinate patches, $(x^1, x^2, \ldots, x^n)$ depend smoothly on $(y^1, y^2, \ldots, y^n)$ and vica versa. Note in particular that two coordinate charts are smoothly compatible if the corresponding coordinate patches are disjoint.

A *smooth atlas* on M is a collection of continuous coordinate systems on M such that the following two conditions hold:—

(i) every point of M belongs to the coordinate patch of at least one of these coordinate systems,

(ii) the coordinate systems in the atlas are smoothly compatible with one another,

Let $\mathcal{A}$ be a smooth atlas on a topological manifold M of dimension n. Let $(u^1, u^2, \ldots, u^n)$ and $(v^1, v^2, \ldots, v^n)$ be continuous coordinate systems, defined over coordinate patches U and V respectively. If the coordinate systems (u^i) and (v^i) are smoothly compatible with all the coordinate systems in the atlas $\mathcal{A}$ then they are smoothly compatible with each other. Indeed suppose that $U \cap V \neq 0$, and let m be a point of $U \cap V$. Then (by condition (i) above) there exists a coordinate system (x^i) belonging to the atlas $\mathcal{A}$ whose coordinate patch includes that point m. But the coordinates (v^i) depend smoothly on the coordinates (x^i), and the coordinates (x^i) depend smoothly on the coordinates (u^i) around m (since the coordinate systems (u^i) and (v^i) are smoothly compatible with all coordinate systems in the atlas $\mathcal{A}$). It follows from the Chain Rule that the coordinates (v^i) depend smoothly on the coordinates (u^i) around m, and similarly the coordinates (u^i) depend smoothly on the coordinates (v^i). Therefore the continuous coordinate systems (u^i) and (v^i) are smoothly compatible with each other.

We deduce that, given a smooth atlas $\mathcal{A}$ on a topological manifold M, we can enlarge $\mathcal{A}$ by adding to to $\mathcal{A}$ all continuous coordinate systems on X that are smoothly compatible with each of the coordinate systems of $\mathcal{A}$. In this way we obtain a smooth atlas on M which is *maximal* in the sense that any coordinate system smoothly compatible with all the charts in the atlas already belongs to the atlas.

Definition A *smooth manifold* $(M, \mathcal{A})$ consists of a topological manifold M together with a maximal smooth atlas $\mathcal{A}$ of coordinate charts on M. A continuous coordinate system $(x^1, x^2, \ldots, x^n)$ on M is said to be *smooth* if and only it belongs to the maximal atlas $\mathcal{A}$. A smooth manifold of dimension 2 is referred as a *smooth surface*.

Example Euclidean space $\mathbb{R}^n$ of dimension n is a smooth manifold. The Cartesian coordinate projections $(x^1, x^2, \ldots, x^n)$ form a coordinate system defined over the whole of $\mathbb{R}^n$. The maximal smooth atlas on $\mathbb{R}^n$ consists of all (curvilinear) coordinate systems smoothly compatible with the Cartesian coordinate system.

Example The unit sphere S^n in $\mathbb{R}^{n+1}$ is a smooth manifold of dimension n. Indeed, for any j between 1 and $n+1$, the restrictions to S^n of the n component functions $x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^{n+1}$ constitute a coordinate system over the coordinate patches U_j^+ and U_j^-, where

$$U_j^+ = \{\mathbf{x} \in S^n : x^j > 0\}, \qquad U_j^- = \{\mathbf{x} \in S^n : x^j < 0\}.$$

The coordinate systems on S^n constructed in this fashion are smooth compatible with each other, and the corresponding coordinate patches cover S^n. Thus if $\mathcal{A}$ is the collection of all continuous coordinate systems on S^n smoothly compatible with the given coordinate systems, then $\mathcal{A}$ is a maximal smooth atlas on S^n. The n-sphere S^n thus becomes a smooth manifold with maximal smooth atlas $\mathcal{A}$.

Let M and N be smooth manifolds of dimension m and n respectively. Let $\varphi: M \to N$ be a continuous function from M to N. Let p_0 be a point of M. The function φ is said to be *smooth* around p_0 if and only if, given any smooth coordinate system $(x^1, x^2, \ldots, x^m)$ around p_0 and any smooth coordinate system $(y^1, y^2, \ldots, y^n)$ around $\varphi(p_0)$, the coordinates $y^1(\varphi(p)), y^2(\varphi(p)), \ldots, y^n(\varphi(p))$

of the image depend smoothly on $(x^1(p), x^2(p), \ldots, x^m(p))$ (i.e., there exist smooth functions $F^1, F^2, \ldots, F^n$ such that

$$y^j(\varphi(p)) = F^j\left(x^1(p), x^2(p), \ldots, x^m(p)\right) \qquad (j = 1, 2, \ldots, n)$$

around p_0. The function $\varphi: M \to N$ is said to be *smooth* if it is smooth around every point of M.

6.2 Partitions of Unity

Let $f: X \to \mathbb{R}$ be a real-valued function defined defined over a topological space X. The *support* $\operatorname{supp} f$ of f is defined to be the closure of the set $\{x \in X : f(x) \neq 0\}$. Thus $\operatorname{supp} f$ is the smallest closed set in X with the property that the function f vanishes on the complement of that set.

Let M be a compact smooth manifold of dimension n, and let $\mathcal{V}$ be an open cover of M. We shall show that there exists a finite collection $f_1, f_2, \ldots, f_k$ of smooth non-negative functions on M such that

(i) $f_1(m) + f_2(m) + f_3(m) + \cdots + f_k(m) = 1$ for all $m \in M$,

(ii) for each function f_i there exists an open set V belonging to $\mathcal{V}$ such that $\operatorname{supp} f_i \subset V$.

A collection $f_1, f_2, \ldots, f_k$ of functions with these properties is referred to a a finite *partition of unity* subordinate to the open cover $\mathcal{V}$.

Lemma 6.1 *Let V be an open set in a smooth manifold M of dimension n, and let v be a point of V. Then there exists a smooth non-negative function $f: M \to \mathbb{R}$ such that $f(v) = 1$ and* $\operatorname{supp} f \subset V$.

Proof First consider the case when V is an open set in $\mathbb{R}^n$. Given $\mathbf{v} \in V$, there exists some $r > 0$ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{v}| < 2r\} \subset V.$$

Define

$$f(\mathbf{x}) = \begin{cases} \exp\left(\dfrac{-|\mathbf{x} - \mathbf{v}|^2}{r^2 - |\mathbf{x} - \mathbf{v}|^2}\right) & \text{if } |\mathbf{x} - \mathbf{v}| < r; \\ 0 & \text{if } |\mathbf{x} - \mathbf{v}| \geq r. \end{cases}$$

Note that $f(\mathbf{x}) = g(1 - r^{-2}|\mathbf{x} - \mathbf{v}|^2)/g(1)$, where $g: \mathbb{R} \to \mathbb{R}$ is defined by

$$g(t) = \begin{cases} \exp(-1/t) & \text{if } t > 0; \\ 0 & \text{if } t \leq 0. \end{cases}$$

Using the fact that $\lim_{u \to +\infty} u^\alpha e^{-u} = 0$ for all real numbers α, one can show that $g: \mathbb{R} \to \mathbb{R}$ is smooth, and that $g^{(j)}(0) = 0$ for all j. Also the function $\mathbf{x} \mapsto |\mathbf{x} - \mathbf{v}|^2$ is smooth on $\mathbb{R}^n$. It follows that $f: \mathbb{R}^n \to \mathbb{R}$ is smooth. Moreover f is non-negative, $f(\mathbf{v}) = 1$ and

$$\operatorname{supp} f = \{\mathbf{x} \in \mathbb{R} : |\mathbf{x} - \mathbf{v}| \leq r\} \subset V.$$

This proves the result when V is a subset of $\mathbb{R}^n$.

Now suppose that V is an open set in a smooth manifold M. Let $v \in V$. Let $x^1, x^2, \ldots, x^n$ be a smooth coordinate system defined over a coordinate patch U, where $v \in U$. If $\varphi: U \to \mathbb{R}^n$ is defined by

$$\varphi(m) = (x^1(m), x^2(m), \ldots, x^n(m))$$

then φ maps U homeomorphically onto an open subset of $\mathbb{R}^n$. Let $W = \varphi(U \cap V)$, and $\mathbf{w} = \varphi(v)$. Then W is an open set in $\mathbb{R}^n$. It follows from the result already proved that there exists a smooth function $F: W \to \mathbb{R}$ such that F is non-negative, $F(\mathbf{w}) = 1$ and $\operatorname{supp} F \subset W$. Define $f: M \to \mathbb{R}$ by

$$f(m) = \begin{cases} F(\varphi(m)) & \text{if } m \in U \cap V; \\ 0 & \text{if } m \notin U \cap V. \end{cases}$$

Then f is a smooth function with the required properties. ∎

Theorem 6.2 *Let M be a compact smooth manifold, and let $\mathcal{V}$ be an open cover of M. Then there exist smooth non-negative functions $f_1, f_2, \ldots, f_k$ with the following properties:—*

(i) $f_1 + f_2 + \cdots + f_k = 1$,

(ii) for each function f_i there exists an open set V belonging to $\mathcal{V}$ such that $\operatorname{supp} f_i \subset V$.

Proof For each point m of M there exists a smooth non-negative function $g_m: M \to \mathbb{R}$ such that $g_m(m) = 1$ and $\operatorname{supp} g_m \subset V$ for at least one open set V belonging to $\mathcal{V}$ (Lemma 6.1). For each $m \in M$, let

$$W_m = \{x \in M : g_m(x) > \tfrac{1}{2}\}.$$

Then $\{W_m : m \in M\}$ is an open cover of M. It follows from the compactness of M that there exists a finite collection $m_1, m_2, \ldots, m_k$ of points of M such that

$$M = W_{m_1} \cup W_{m_2} \cup \cdots \cup W_{m_k}.$$

Set $f_i(x) = g_{m_i}(x)/G(x)$ for all $x \in M$, where

$$G(x) = g_{m_1}(x) + g_{m_2}(x) + \cdots g_{m_k}(x).$$

Then $f_1, f_2, \ldots, f_k$ is a collection of smooth functions on M with the required properties. ∎

Remark There is a generalization of Theorem 6.2 applicable to non-compact smooth manifolds. Let M be a smooth manifold (not necessarily compact). A collection $\{f_i : i \in I\}$ of functions on M is said to be *locally finite* if and only if, given any $m \in M$ there exists an open set U containing M such that only finitely many of the functions f_i take one non-zero values on U. If $\{f_i : i \in I\}$ is a locally finite collection of smooth functions on M then the sum $\sum_{i \in I} f_i$ is a well-defined smooth function on M. A *locally finite partition of unity* on M is a locally finite collection $\{f_i : i \in I\}$ of smooth non-negative functions on M such that $\sum_{i \in I} f_i = 1$. It can be shown that there exists a locally finite partition of unity subordinate to any open cover of M.