4 Compact Topological Spaces

4.1 Open Covers and Compactness

Let $X$ be a topological space, and let $A$ be a subset of $X$. A collection of open sets in $X$ is said to cover $A$ if and only if every point of $A$ belongs to at least one of these open sets. In particular, an open cover of $X$ is collection of open sets in $X$ that covers $X$.

If $U$ and $V$ are open covers of some topological space $X$ then $V$ is said to be a subcover of $U$ if and only if every open set in $V$ belongs to $U$.

**Definition** A topological space $X$ is said to be compact if and only if every open cover of $X$ possesses a finite subcover.

**Lemma 4.1** Let $X$ be a topological space. A subset $A$ of $X$ is compact (with respect to the subspace topology on $A$) if and only if, given any collection $U$ of open sets in $X$ covering $A$, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $U$ such that

$$A \subset V_1 \cup V_2 \cup \cdots \cup V_r.$$ 

**Proof** If $U$ is any collection of open sets in $X$ covering $A$ then $\{V \cap A : V \in U\}$ is an open cover of $A$. Moreover it follows from the definition of the subspace topology on $A$ that, given any open cover $U_A$ of $A$ there exists some collection $U$ of open sets in $X$ such that

$$U_A = \{V \cap A : V \in U\}.$$

The required result now follows directly from the definition of compactness.

4.2 The Heine-Borel Theorem

We now show that any closed bounded interval in the real line is compact. This result is known as the Heine-Borel Theorem. The proof of this theorem uses the least upper bound principle which states that, given any non-empty set $S$ of real numbers which is bounded above, there exists a least upper bound (or supremum) $\sup S$ for the set $S$.

**Theorem 4.2 (Heine Borel)** Let $a$ and $b$ be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$. 

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Proof Let $\mathcal{U}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let $S$ be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by finitely many of the open sets belonging to $\mathcal{U}$. Let $s = \sup S$. Now $s \in W$ for some open set $W$ belonging to $\mathcal{U}$. But then there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$ (since $W$ is open in $\mathbb{R}$). Also there exists some $\tau \in S$ satisfying $\tau > s - \delta$ (since $s - \delta$ is not an upper bound for the set $S$). Now if $V_1, V_2, \ldots, V_r$ is any finite collection of open sets belonging to $\mathcal{U}$ which cover $[a, \tau]$ then

$$[a, t] \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W$$

for all $t \in [a, b]$ satisfying $s \leq t < s + \delta$. Thus $t \in S$ for all $t$ satisfying $a \leq t \leq b$ and $s \leq t < s + \delta$. In particular $s \in S$. Moreover $s = b$, since otherwise $s$ would not be an upper bound of the set $S$. Thus $b \in S$. This means that the interval $[a, b]$ can be covered by finitely many open sets belonging to $\mathcal{U}$, as required.

4.3 Basic Properties of Compact Topological Spaces

Lemma 4.3 Let $X$ be a compact topological space, and let $A$ be a closed subset of $X$. Then $A$ is compact.

Proof Let $\mathcal{U}$ be a collection of open sets in $X$ covering $A$. If we adjoin the open set $X \setminus A$ to the collection $\mathcal{U}$ then we obtain an open cover of the space $X$. This open cover possesses a finite subcover, since $X$ is compact. In particular, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to the collection $\mathcal{U}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$. Thus $A$ is compact, by Lemma 4.1.

Lemma 4.4 Let $X$ and $Y$ are topological spaces, and $f: X \to Y$ be a continuous function. Let $A$ be a compact subset of $X$. Then $f(A)$ is a compact subset of $Y$.

Proof Let $\mathcal{V}$ be a collection of open sets in $Y$ which covers $f(A)$, and let $\mathcal{W}$ be the collection of open sets in $X$ consisting of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. Then $\mathcal{W}$ covers $A$. It follows from the compactness of $A$ that there exist open sets $V_1, V_2, \ldots, V_r$ belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_r).$$

But then

$$f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_r.$$
Thus $f(A)$ is compact. 

**Lemma 4.5** Let $X$ be a compact topological space and let $f: X \to \mathbb{R}$ be a continuous function from $X$ to $\mathbb{R}$. Then $f$ is bounded above and below on $X$.

**Proof** It follows from Lemma 4.4 that the image $f(X)$ of the function $f$ is a compact subset of $\mathbb{R}$. Let $U_1, U_2, U_3, \ldots$ be the open subsets of $\mathbb{R}$ defined by $U_m = \{t \in \mathbb{R} : -m < t < m\}$ for all natural numbers $m$. Then the collection $\{U_m : m \in \mathbb{N}\}$ of open sets covers $\mathbb{R}$. It follows from the compactness of $f(X)$ that $f(X)$ can be covered by finitely many of these open sets. Suppose that

$$f(X) \subseteq U_{m_1} \cup U_{m_2} \cup \cdots \cup U_{m_r},$$

where $m_1 < m_2 < \cdots < m_r$. Then $f(X) \subseteq U_{m_r}$ (since $U_{m_j} \subset U_{m_r}$ for all $j$ satisfying $j < r$). We deduce that $-m_r < f(x) < m_r$ for all $x \in X$. Thus the function $f$ is bounded above and below on $X$.

**Proposition 4.6** Let $X$ be a compact topological space and let $f: X \to \mathbb{R}$ be a continuous real-valued function on $X$. Then there exist points $u$ and $v$ of $X$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.

**Proof** Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. If $f(x) < M$ for all $x \in X$ then the function $g: X \to \mathbb{R}$ defined by $g(x) = 1/(M - f(x))$ would be a continuous function on $X$ that was not bounded above, contradicting Lemma 4.5. Therefore there must exist $v \in X$ for which $f(v) = M$. Similarly if $f(x) > m$ for all $x \in X$ then the function $h: X \to \mathbb{R}$ defined by $h(x) = 1/(f(x) - m)$ would be a continuous function on $X$ that was not bounded above, again contradicting Lemma 4.5. Therefore there must exist $u \in X$ for which $f(u) = m$. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required.

### 4.4 Compact Hausdorff Spaces

**Proposition 4.7** Let $X$ be a Hausdorff topological space, and let $K$ be a compact subset of $X$. Let $x$ be a point of $X \setminus K$. Then there exist open subsets $V_x$ and $W_x$ of $X$ such that $x \in V_x$, $K \subset W_x$ and $V_x \cap W_x = \emptyset$.

**Proof** Let $x$ be a point of $X \setminus K$. For each point $y$ of $K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since $X$ is a Hausdorff space). But it then follows from the compactness of $K$ that there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of $K$ such that

$$K \subset W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$
Define
\[ V_x = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \quad W_x = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}. \]

Then \( V_x \) and \( W_x \) are open sets, \( x \) belongs to \( V_x \), \( K \subset W_x \) and \( V_x \cap W_x = \emptyset \), as required.

**Corollary 4.8** Let \( X \) be a Hausdorff topological space, and let \( K \) be a compact subset of \( X \). Then \( K \) is closed.

**Proof** It follows immediately from Proposition 4.7 that, for each point \( x \) of \( X \setminus K \), there exists an open set \( V_x \) such that \( x \in V_x \) and \( V_x \cap K = \emptyset \). But then \( X \setminus K \) is equal to the union of the open sets \( V_x \) as \( x \) ranges over all points of \( X \setminus K \). But any set that is a union of open sets is itself an open set. We conclude that \( X \setminus K \) is open. Thus \( K \) is closed.

**Proposition 4.9** Let \( X \) be a Hausdorff topological space, and let \( K_1 \) and \( K_2 \) be compact subsets of \( X \), where \( K_1 \cap K_2 = \emptyset \). Then there exist open sets \( U_1 \) and \( U_2 \) such that \( K_1 \subset U_1 \), \( K_2 \subset U_2 \) and \( U_1 \cap U_2 = \emptyset \).

**Proof** It follows from Proposition 4.7 that, for each point \( x \) of \( K_1 \), there exist open sets \( V_x \) and \( W_x \) such that \( x \in V_x \), \( K_2 \subset W_x \) and \( V_x \cap W_x = \emptyset \). But it then follows from the compactness of \( K_1 \) that there exists a finite set \( \{x_1, x_2, \ldots, x_r\} \) of points of \( K_1 \) such that
\[ K_1 \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}. \]

Define
\[ U_1 = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}, \quad U_2 = W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x_r}. \]

Then \( U_1 \) and \( U_2 \) are open sets, \( K_1 \subset U_1 \), \( K_2 \subset U_2 \) and \( U_1 \cap U_2 = \emptyset \), as required.

**Remark** A topological space \( X \) is said to be normal if and only if, given any closed subsets \( F_1 \) and \( F_2 \) of \( X \) for which \( F_1 \cap F_2 = \emptyset \), there exist open sets \( U_1 \) and \( U_2 \) of \( X \) for which \( U_1 \cap U_2 = \emptyset \). Now every closed subset of a compact topological space is compact, by Lemma 4.3. It follows from Proposition 4.9 that every compact Hausdorff space is normal.

**Lemma 4.10** Let \( X \) be a compact topological space, let \( Y \) be a Hausdorff space, and let \( f: X \to Y \) be a continuous function from \( X \) to \( Y \). Then \( f(K) \) is closed in \( Y \) for every closed set \( K \) in \( X \).
Proof Let $K$ be a closed subset of $X$. Every closed subset of a compact topological space is compact, by Lemma 4.3. Therefore $K$ is compact. It then follows from Lemma 4.4 that $f(K)$ is compact. But then $f(K)$ is closed, by Corollary 4.8, since $Y$ is Hausdorff. Thus $f(K)$ is closed in $Y$ for every closed set $K$ in $X$. □

Theorem 4.11 Let $X$ be a compact topological space, let $Y$ be a Hausdorff space, and let $f:X \to Y$ be a continuous function from $X$ to $Y$ which is also a bijection (i.e., $f$ is both one-to-one and onto). Then $f:X \to Y$ is a homeomorphism.

Proof The function $f$ is invertible, since it is a bijection. Let $g:Y \to X$ be the inverse of $f:X \to Y$. Let $U$ be an open set in $X$. Then $X \setminus U$ is closed in $X$, and hence $f(X \setminus U)$ is closed in $Y$, by Lemma 4.10. But

$$f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U).$$

It follows that $g^{-1}(U)$ is open in $Y$. Thus $g:Y \to X$ is continuous. We deduce that $f:X \to Y$ is a homeomorphism, as required. □

We recall that a function $f:X \to Y$ from a topological space $X$ to a topological space $Y$ is said to be an identification map if it is surjective and satisfies the following condition: a subset $U$ of $Y$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$.

Proposition 4.12 Let $X$ be a compact topological space, let $Y$ be a Hausdorff space. If $f:X \to Y$ is a continuous surjection then $f$ is an identification map.

Proof The function $f:X \to Y$ is surjective, and $f^{-1}(U)$ is open in $X$ for any open subset $U$ of $Y$, since $f$ is continuous. Thus, to prove that $f:X \to Y$ is an identification map, it only remains to show that if $U$ is a subset of $Y$ such that $f^{-1}(U)$ is open in $X$ then $U$ is open in $Y$.

Let $K = X \setminus f^{-1}(U)$. If $f^{-1}(U)$ is open in $X$ then $K$ is closed. It follows from Lemma 4.10 that $f(K)$ is closed in $Y$.

Now if $x \in K$ then $x \notin f^{-1}(U)$, and hence $f(x) \in Y \setminus U$. Thus $f(K) \subset Y \setminus U$. But if $y$ is any point of $Y \setminus U$ then $y = f(x)$ for some $x \in X$, since $f$ is surjective, and moreover $x \in K$ (since $f(x) \notin U$). Thus $f(K) = Y \setminus U$. But $f(K)$ is closed in $Y$. It follows that $U$ is open in $Y$, as required. □

Example Let $S^1$ be the unit circle in $\mathbb{R}^2$, defined by

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$
Let \( q : [0, 1] \to S^1 \) be the function defined by
\[
q(t) = (\cos 2\pi t, \sin 2\pi t) \quad (t \in [0, 1]).
\]
The function \( q \) is surjective. Moreover the closed interval \([0, 1]\) is compact, and the circle \( S^1 \) is Hausdorff. Therefore the function \( q : [0, 1] \to S^1 \) is an identification map. Thus a function \( f : S^1 \to Z \) from the circle \( S^1 \) to some topological space \( Z \) is continuous if and only if the composition function \( f \circ q : [0, 1] \to Z \) is continuous (see Lemma 2.19).

### 4.5 Finite Products of Compact Spaces

We shall show that a finite Cartesian product of compact spaces is compact. To prove this, we apply the following result, known as the Tube Lemma.

**Lemma 4.13** Let \( X \) and \( Y \) be topological spaces, let \( K \) be a compact subset of \( Y \), and \( U \) be an open set in \( X \times Y \). Let \( V \) be the subset of \( X \) defined by
\[
V = \{ x \in X : \{ x \} \times K \subset U \}.
\]
Then \( V \) is an open set in \( X \).

**Proof** Let \( x \) be a point of \( V \). For each point \( y \) of \( K \) there exist open subsets \( D_y \) and \( E_y \) of \( X \) and \( Y \) respectively such that \((x, y) \in D_y \times E_y \) and \( D_y \times E_y \subset U \). But \( K \) is compact. Therefore there exists a finite set \( \{y_1, y_2, \ldots, y_k\} \) of points of \( K \) such that
\[
K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k}.
\]
Set
\[
N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}.
\]
Then \( N_x \) is an open set in \( X \). Moreover
\[
N_x \times K \subset \bigcup_{i=1}^{k} (D_{y_i} \times E_{y_i}) \subset U,
\]
so that \( N_x \subset V \). It follows that \( V \) is the union of the open sets \( N_x \) for all \( x \in V \). Thus \( V \) is itself an open set in \( X \), as required. \( \square \)

**Theorem 4.14** Let \( X \) and \( Y \) be compact topological spaces. Then \( X \times Y \) is compact.
Proof Let $\mathcal{U}$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

Let $x$ be a point of $X$. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, hence there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $\mathcal{U}$ such that

$$\{x\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r.$$ 

Let

$$V_x = \{x' \in X : \{x'\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r\}.$$ 

It follows from Lemma 4.13 that $V_x$ is an open set in $X$. We have therefore shown that, for each point $x$ in $X$, there exists an open set $V_x$ in $X$ containing the point $x$ such that $V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $\mathcal{U}$.

Now $\{V_x : x \in X\}$ is an open cover of the compact space $X$. This cover possesses a finite subcover. Thus there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of $X$ such that

$$X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}.$$ 

It follows from this that $X \times Y$ can be covered by finitely many open sets belonging to the open cover $\mathcal{U}$ (since $X \times Y$ is a finite union of sets of the form $V_x \times Y$, and each of these sets can be covered by finitely many of the open sets belonging to $\mathcal{U}$). Therefore $X \times Y$ is compact. 

Corollary 4.15 Let $X_1, X_2, \ldots, X_n$ be compact topological spaces. Then the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is compact.

Proof It follows easily from the definition of the product topology that the product topologies on $X_1 \times X_2 \times \cdots \times X_n$ and $(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$ coincide. The desired result therefore follows from Theorem 4.14 by induction on $n$.

Theorem 4.16 Let $K$ be a subset of $\mathbb{R}^n$. Then $K$ is compact if and only if $K$ is both closed and bounded.

Proof Suppose that $K$ is compact. We show that $K$ is closed and bounded. Note that $K$ is closed, since $\mathbb{R}^n$ is Hausdorff, and all compact subsets of Hausdorff spaces are closed, by Corollary 4.8. Consider the open cover of $\mathbb{R}^n$ provided by the sets $U_m$ for all positive integers $m$, where

$$U_m = \{x \in \mathbb{R}^n : |x| < m\}.$$ 

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The set $K$ must be covered by finitely many of these open sets, since $K$ is compact. Suppose that

$$K \subset U_{m_1} \cup U_{m_2} \cup \cdots \cup U_{m_r},$$

where $m_1 < m_2 < \cdots < m_r$. Then $K \subset U_{m_r}$, and hence $|x| < m_r$ for all $x \in K$. Thus $K$ is bounded.

Conversely suppose that $K$ is both closed and bounded. Then there exists some real number $L$ such that $K$ is contained within the closed cube $C$ given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$ 

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 4.2), and $C$ is the Cartesian product of $n$-copies of the compact set $[-L, L]$. It follows from Corollary 4.15 that $C$ is compact. But $K$ is a closed subset of $C$, and a closed subset of a compact topological space is itself compact, by Lemma 4.3. Thus $K$ is compact, as required. 

### 4.6 Norms on Vector Spaces

Let $V$ be a vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|$ on $V$ is a function sending an element $v$ of $V$ to some real number, denoted by $\|v\|$, which satisfies the following properties:—

(i) $\|v\| \geq 0$ for all $v \in V$,

(ii) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$,

(iii) $\|\lambda v\| = |\lambda| \|v\|$ for all $v \in V$ and $\lambda \in \mathbb{F}$,

(iv) $\|v\| = 0$ if and only if $v = 0$.

**Example** Let $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ be the functions from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\|x\|_1 = \sum_{j=1}^n |x_j|,$$

$$\|x\|_2 = \left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}},$$

$$\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|),$$

for each $x \in \mathbb{R}^n$, where $x = (x_1, x_2, \ldots, x_n)$. Then $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$. (Note that $\|\cdot\|_2$ is the standard Euclidean norm on $\mathbb{R}^n$, which we have denoted by $|\cdot|$.)
A norm \( \| \cdot \| \) on a vector space \( V \) induces a corresponding distance function on \( V \): the distance \( d(v, w) \) between elements \( v \) and \( w \) of \( V \) is defined by \( d(v, w) = \| v - w \| \). This distance function satisfies the metric space axioms. Thus any vector space with a norm can be regarded as a metric space. The distance function in turn induces a topology on \( V \): a subset \( U \) of \( V \) is open (with respect to the topology induced by the norm \( \| \cdot \| \)) if and only if, given any point \( u \) of \( U \), there exists some \( \delta > 0 \) such that

\[
\{ v \in V : \| v - u \| < \delta \} \subset U.
\]

Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be norms on the vector space \( V \). The norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are said to be equivalent if and only if there exist constants \( c \) and \( C \), where \( 0 < c \leq C \), such that

\[
c \| v \|_1 \leq \| v \|_2 \leq C \| v \|_1
\]

for all \( v \in V \).

**Lemma 4.17** Let \( V \) be a real (or complex) vector space, and let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be norms on \( V \). The norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) generate the same topology on \( V \) if and only if they are equivalent.

**Proof** Suppose that the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are equivalent. Then there exist constants \( c \) and \( C \), where \( 0 < c \leq C \), such that \( c \| v \|_1 \leq \| v \|_2 \leq C \| v \|_1 \) for all \( v \in V \). Thus, for any point \( u \) of \( V \),

\[
\begin{align*}
\{ v \in V : \| v - u \|_2 < \delta \} & \subset \{ v \in V : \| v - u \|_1 < \delta/c \}, \\
\{ v \in V : \| v - u \|_1 < \delta \} & \subset \{ v \in V : \| v - u \|_2 < C\delta \}.
\end{align*}
\]

Let \( U \) be a subset of \( V \). If \( U \) is open with respect to the topology induced by the norm \( \| \cdot \|_1 \) then, given any point \( u \) of \( U \), there exists some \( \delta > 0 \) such that

\[
\{ v \in V : \| v - u \|_1 < \delta/c \} \subset U.
\]

But then

\[
\{ v \in V : \| v - u \|_2 < \delta \} \subset \{ v \in V : \| v - u \|_1 < \delta/c \} \subset U,
\]

so that \( U \) is open with respect to the topology induced by the norm \( \| \cdot \|_2 \). A similar proof, using the fact that

\[
\{ v \in V : \| v - u \|_1 < \delta \} \subset \{ v \in V : \| v - u \|_2 < C\delta \}
\]
for all $\delta > 0$, shows that if $U$ is open with respect to the topology induced by the norm $\| \cdot \|_2$ then $U$ is open with respect to the topology induced by the norm $\| \cdot \|_1$. Thus equivalent norms induce the same topology on $V$.

Conversely suppose that $\| \cdot \|_1$ and $\| \cdot \|_2$ are norms on $V$ which induce the same topology on $V$. Now $\{ v \in V : \| v \|_2 < 1 \}$ is open with respect to the topology induced by $\| \cdot \|_2$. This set is therefore open with respect to the topology induced by $\| \cdot \|_1$, and hence there exists some $\delta_1 > 0$ such that

$$\{ v \in V : \| v \|_1 < \delta_1 \} \subset \{ v \in V : \| v \|_2 < 1 \}.$$ 

A similar argument shows that there exists $\delta_2 > 0$ such that

$$\{ v \in V : \| v \|_2 < \delta_2 \} \subset \{ v \in V : \| v \|_1 < 1 \}.$$ 

Set $C = 2/\delta_1$ and $c = \delta_2/2$. Let $v$ be a non-zero element of $V$, and let $\lambda_v = \delta_1/2\| v \|_1$. Then

$$\| \lambda_v v \|_1 = |\lambda_v|\| v \|_1 = \frac{1}{2}\delta_1 < \delta_1$$

and hence $\| \lambda_v v \|_2 < 1$. But

$$\| \lambda_v v \|_2 = |\lambda_v|\| v \|_2 = \frac{\| v \|_2}{C\| v \|_1}.$$ 

Thus $\| v \|_2 \leq C\| v \|_1$. A similar argument shows that $\| v \|_1 \leq c^{-1}\| v \|_2$. Thus $c\| v \|_1 \leq \| v \|_2 \leq C\| v \|_1$ for all non-zero elements $v$ of $V$. These inequalities also hold if $v = 0$. We conclude that if the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ induce the same topology on $V$ then these norms are equivalent.

We shall show that any two norms on $\mathbb{R}^n$ are equivalent (Theorem 4.19). Since any $n$-dimensional real vector space is isomorphic to $\mathbb{R}^n$ for all natural numbers $n$, this shows that any two norms on a finite-dimensional real vector space are equivalent, and thus generate the same topology on that vector space. (This result does not apply to infinite-dimensional vector spaces.)

**Lemma 4.18** Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$. Then the function $x \mapsto \| x \|$ is continuous with respect to the usual topology on $\mathbb{R}^n$ (i.e., the topology on $\mathbb{R}^n$ induced by the Euclidean norm).

**Proof** Let $e_1, e_2, \ldots, e_n$ denote the basis of $\mathbb{R}^n$ given by

$$e_1 = (1, 0, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \quad \ldots \quad e_n = (0, 0, 0, \ldots, 1).$$
Let \(x\) and \(y\) be points of \(\mathbb{R}^n\), where
\[
x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n).
\]

Now
\[
||x|| - ||y|| \leq ||x - y||
\]
since
\[
||x|| \leq ||x - y|| + ||y||, \quad ||y|| \leq ||x - y|| + ||x||.
\]

Also
\[
||x - y|| = \left( \sum_{j=1}^{n} (x_j - y_j)^2 \right)^{1/2} \leq \sum_{j=1}^{n} |x_j - y_j| \cdot ||e_j||.
\]

Let \(c_j = ||e_j||\) for \(j = 1, 2, \ldots, n\). It follows from Schwarz’ Inequality (Lemma 1.1) that
\[
\sum_{j=1}^{n} |x_j - y_j| \cdot ||e_j|| = \sum_{j=1}^{n} |x_j - y_j| \cdot c_j \leq \left( \sum_{j=1}^{n} (x_j - y_j)^2 \right)^{1/2} \left( \sum_{j=1}^{n} c_j^2 \right)^{1/2} = C||x - y||,
\]
where \(C^2 = c_1^2 + c_2^2 + \cdots + c_n^2\), and where \(||x - y||\) denotes the Euclidean norm of \(x - y\). We conclude therefore that
\[
||x|| - ||y|| \leq ||x - y||
\]

This shows that the function from \(\mathbb{R}^n\) to \(\mathbb{R}\) given by \(x \mapsto ||x||\) is continuous on \(\mathbb{R}^n\) with respect to the usual topology on \(\mathbb{R}^n\).

**Theorem 4.19** Any two norms on \(\mathbb{R}^n\) are equivalent, and therefore induce the same topology on \(\mathbb{R}^n\). (This topology is the usual topology on \(\mathbb{R}^n\).)

**Proof** Let \(||.||_1\) be any norm on \(\mathbb{R}^n\). We show that \(||.||_1\) is equivalent to the Euclidean norm \(|.|\). Let \(S^{n-1}\) denote the unit sphere in \(\mathbb{R}^n\), defined by
\[
S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.
\]

Now \(S^{n-1}\) is a compact subset of \(\mathbb{R}^n\), by Theorem 4.16, since it is both closed and bounded. Also the function \(f: S^{n-1} \to \mathbb{R}\) defined by \(f(x) = ||x||_1\) is continuous, by Lemma 4.18. Note that \(f(x) > 0\) for all \(x \in S^{n-1}\) (see properties \((i)\) and \((iv)\) in the definition of norms). It follows from Proposition 4.6 that there exist points \(u\) and \(v\) of \(S^{n-1}\) such that \(f(u) \leq f(x) \leq f(v)\) for all \(x \in S^{n-1}\). Set \(c_1 = f(u) = ||u||_1\) and \(C_1 = f(v) = ||v||_1\). Then \(0 < c_1 \leq C_1\).
If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $(1/|\mathbf{x}|)\mathbf{x}$ is an element of $S^{n-1}$, and hence

$$c_1 \leq \left\| \frac{\mathbf{x}}{|\mathbf{x}|} \right\|_1 \leq C_1.$$ 

Thus $c_1|x| \leq \|x\|_1 \leq C_1|x|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|\cdot\|_1$ is equivalent to the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$.

If $\|\cdot\|_2$ is any other norm on $\mathbb{R}^n$ then $\|\cdot\|_2$ is also equivalent to the Euclidean norm on $\mathbb{R}^n$, and hence there exist constants $c_2$ and $C_2$ satisfying $0 < c_2 \leq C_2$ such that $c_2|x| \leq \|x\|_2 \leq C_2|x|$ for all $\mathbf{x} \in \mathbb{R}^n$. But then

$$\frac{c_2}{C_1}\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq \frac{C_2}{c_1}\|\mathbf{x}\|_1.$$ 

Thus the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. This shows that any two norms on $\mathbb{R}^n$ are equivalent. It then follows from Lemma 4.17 that any two norms on $\mathbb{R}^n$ generate the same topology on $\mathbb{R}^n$. 

### 4.7 The Lebesgue Lemma

**Definition** Let $(X, d)$ be a metric space, and let $A$ be a subset of $X$. The *diameter* of the set $A$ is defined to be the supremum

$$\sup_{u, v \in A} d(u, v)$$

of the distance from the point $u$ to the point $v$ as $u$ and $v$ range over all the points of the set $A$. (If the distance $d(u, v)$ from $u$ to $v$ is not bounded above as $u$ and $v$ range over the set $A$ then the diameter of $A$ is defined to be $+\infty$.)

We now state and prove the *Lebesgue Lemma*.

**Lemma 4.20 (Lebesgue Lemma)** Let $(X, d)$ be a compact metric space. Let $\mathcal{U}$ be an open cover of $X$. Then there exists a positive real number $\delta$ such that every subset of $X$ whose diameter is less than $\delta$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.

**Proof** Every point of $X$ is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point $x$ of $X$, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point $x$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius $\delta_x$ about the points $x$ of $X$ forms an open cover of the compact
space $X$. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of $X$ such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta > 0$ be given by

$$\delta = \min(\delta_1, \delta_2, \ldots, \delta_r).$$

Suppose that $A$ is a subset of $X$ whose diameter is less than $\delta$. Let $u$ be a point of $A$. Then $u$ belongs to $B(x_i, \delta_i)$ for some integer $i$ between 1 and $r$. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point $v$ of $A$,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $U$. Thus $A$ is contained wholly within one of the open sets belonging to $U$, as required.

Let $U$ be an open cover of a compact metric space $X$. A Lebesgue number for the open cover $U$ is a positive real number $\delta$ such that every subset of $X$ whose diameter is less than $\delta$ is contained wholly within one of the open sets belonging to the open cover $U$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

The following result follows from the Heine-Borel Theorem (Theorem 4.2) and the Lebesgue Lemma (Lemma 4.20).

**Theorem 4.21** Let $X$ be a topological space, and let $U$ be an open cover of $X$. Let $a$ and $b$ be real numbers satisfying $a < b$, and let $\gamma : [a, b] \to X$ be a continuous function from the closed bounded interval $[a, b]$ into $X$. Then there exist $t_0, t_1, \ldots, t_r \in [a, b]$, where

$$a = t_0 < t_1 < t_2 < \cdots < t_r = b,$$

such that, for each $i$, $\gamma([t_{i-1}, t_i])$ is contained wholly within one of the open sets belonging to the open cover $U$.

**Proof** Let $V$ be the open cover of $[a, b]$ consisting of all the subsets of $[a, b]$ that are of the form $\gamma^{-1}(U)$ for some open set $U$ belonging to $U$. The closed bounded interval $[a, b]$ is a compact metric space. Let $\delta > 0$ be a Lebesgue number for this open cover. Choose $t_0, t_1, \ldots, t_r$ such that $t_0 = a$, $t_r = b$ and $0 < t_i - t_{i-1} < \delta$ for $i = 1, 2, \ldots, r$. Then, for each $i$, $[t_{i-1}, t_i] \subset \gamma^{-1}(U)$ and that $\gamma([t_{i-1}, t_i]) \subset U$ for some open set $U$ belonging to the open cover $U$. 


Let $X$ and $Y$ be metric spaces with distance functions $d_X$ and $d_Y$ respectively, and let $f: X \rightarrow Y$ be a function from $X$ to $Y$. The function $f$ is said to be uniformly continuous on $X$ if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points $x$ and $x'$ of $X$ satisfying $d_X(x, x') < \delta$. (The value of $\delta$ should be independent of both $x$ and $x'$.)

**Theorem 4.22** Let $X$ and $Y$ be metric spaces. Suppose that $X$ is compact. Then every continuous function from $X$ to $Y$ is uniformly continuous.

**Proof** Let $d_X$ and $d_Y$ denote the distance functions for the metric spaces $X$ and $Y$ respectively. Let $f: X \rightarrow Y$ be a continuous function from $X$ to $Y$. We must show that $f$ is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \frac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}(B_Y(y, \frac{1}{2}\varepsilon))$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about $y$ in $Y$. Now $B_Y(y, \frac{1}{2}\varepsilon)$ is open in $Y$ (see Lemma 1.5), and $f$ is continuous. Therefore $V_y$ is open in $X$ for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space $X$. It follows from the Lebesgue Lemma (Lemma 4.20) that there exists some $\delta > 0$ such that every subset of $X$ whose diameter is less than $\delta$ is a subset of some set $V_y$. Let $x$ and $x'$ be points of $X$ satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$ which is less than $\delta$. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f: X \rightarrow Y$ is uniformly continuous, as required.  

Let $K$ be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 4.16 and Theorem 4.22 that any continuous function $f: K \rightarrow \mathbb{R}^k$ is uniformly continuous.