Course 212: Academic Year 1989-1990
Section 4: Compact Topological Spaces

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Contents

4 Compact Topological Spaces

4.1 Open Covers and Compactness

Let X be a topological space, and let A be a subset of X. A collection of open sets in X is said to *cover* A if and only if every point of A belongs to at least one of these open sets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set in $\mathcal{V}$ belongs to $\mathcal{U}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 4.1 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{U}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that*

$$A \subset V_1 \cup V_2 \cup \cdots \cup V_r.$$

Proof If $\mathcal{U}$ is any collection of open sets in X covering A then $\{V \cap A : V \in \mathcal{U}\}$ is an open cover of A. Moreover it follows from the definition of the subspace topology on A that, given any open cover $\mathcal{U}_A$ of A there exists some collection $\mathcal{U}$ of open sets in X such that

$$\mathcal{U}_A = \{V \cap A : V \in \mathcal{U}\}.$$

The required result now follows directly from the definition of compactness. ∎

4.2 The Heine-Borel Theorem

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem*. The proof of this theorem uses the *least upper bound principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 4.2 (Heine Borel) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{U}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by finitely many of the open sets belonging to $\mathcal{U}$. Let $s = \sup S$. Now $s \in W$ for some open set W belonging to $\mathcal{U}$. But then there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$ (since W is open in $\mathbb{R}$). Also there exists some $\tau \in S$ satisfying $\tau > s - \delta$ (since $s - \delta$ is not an upper bound for the set S). Now if $V_1, V_2, \ldots, V_r$ is any finite collection of open sets belonging to $\mathcal{U}$ which cover $[a, \tau]$ then

$$[a, t] \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W$$

for all $t \in [a, b]$ satisfying $s \leq t < s + \delta$. Thus $t \in S$ for all t satisfying $a \leq t \leq b$ and $s \leq t < s + \delta$. In particular $s \in S$. Moreover $s = b$, since otherwise s would not be an upper bound of the set s. Thus $b \in S$. This means that the interval $[a, b]$ can be covered by finitely many open sets belonging to $\mathcal{U}$, as required. ∎

4.3 Basic Properties of Compact Topological Spaces

Lemma 4.3 *Let X be a compact topological space, and let A be a closed subset of X. Then A is compact.*

Proof Let $\mathcal{U}$ be a collection of open sets in X covering A. If we adjoin the open set $X \setminus A$ to the collection $\mathcal{U}$ then we obtain an open cover of the space X. This open cover possesses a finite subcover, since X is compact. In particular, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to the collection $\mathcal{U}$ such that $A \subset V_1 \cup V_2 \cup V_r$. Thus A is compact, by Lemma 4.1. ∎

Lemma 4.4 *Let X and Y are topological spaces, and $f\colon X \to Y$ be a continuous function. Let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$, and let $\mathcal{W}$ be the collection of open sets in X consisting of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. Then $\mathcal{W}$ covers A. It follows from the compactness of A that there exist open sets $V_1, V_2, \ldots, V_r$ belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_r).$$

But then

$$f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_r.$$

Thus $f(A)$ is compact. ∎

Lemma 4.5 *Let X be a compact topological space and let $f: X \to \mathbb{R}$ be a continuous function from X to $\mathbb{R}$. Then f is bounded above and below on X.*

Proof It follows from Lemma 4.4 that the image $f(X)$ of the function f is a compact subset of $\mathbb{R}$. Let $U_1, U_2, U_3, \ldots$ be the open subsets of $\mathbb{R}$ defined by $U_m = \{t \in \mathbb{R} : -m < t < m\}$ for all natural numbers m. Then the collection $\{U_m : m \in \mathbb{N}\}$ of open sets covers $\mathbb{R}$. It follows from the compactness of $f(X)$ that $f(X)$ can be covered by finitely many of these open sets. Suppose that

$$f(X) \subset U_{m_1} \cup U_{m_2} \cup \cdots \cup U_{m_r},$$

where $m_1 < m_2 < \cdots < m_r$. Then $f(X) \subset U_{m_r}$ (since $U_{m_j} \subset U_{m_r}$ for all j satisfying $j < r$). We deduce that $-m_r < f(x) < m_r$ for all $x \in X$. Thus the function f is bounded above and below on X. ∎

Proposition 4.6 *Let X be a compact topological space and let $f: X \to \mathbb{R}$ be a continuous real-valued function on X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. If $f(x) < M$ for all $x \in X$ then the function $g: X \to \mathbb{R}$ defined by $g(x) = 1/(M - f(x))$ would be a continuous function on X that was not bounded above, contradicting Lemma 4.5. Therefore there must exist $v \in X$ for which $f(v) = M$. Similarly if $f(x) > m$ for all $x \in X$ then the function $h: X \to \mathbb{R}$ defined by $h(x) = 1/(f(x) - m)$ would be a continuous function on X that was not bounded above, again contradicting Lemma 4.5. Therefore there must exist $u \in X$ for which $f(u) = m$. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

4.4 Compact Hausdorff Spaces

Proposition 4.7 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Then there exist open subsets V_x and W_x of X such that $x \in V_x$, $K \subset W_x$ and $V_x \cap W_x = \emptyset$.*

Proof Let x be a point of $X \setminus K$. For each point y of K there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But it then follows from the compactness of K that there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that

$$K \subset W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Define

$$V_x = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W_x = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V_x and W_x are open sets, x belongs to V_x, $K \subset W_x$ and $V_x \cap W_x = \emptyset$, as required. ∎

Corollary 4.8 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Then K is closed.*

Proof It follows immediately from Proposition 4.7 that, for each point x of $X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$. But any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open. Thus K is closed. ∎

Proposition 4.9 *Let X be a Hausdorff topological space, and let K_1 and K_2 be compact subsets of X, where $K_1 \cap K_2 = \emptyset$. Then there exist open sets U_1 and U_2 such that $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.*

Proof It follows from Proposition 4.7 that, for each point x of K_1, there exist open sets V_x and W_x such that $x \in V_x$, $K_2 \subset W_x$ and $V_x \cap W_x = \emptyset$. But it then follows from the compactness of K_1 that there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of K_1 such that

$$K_1 \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}.$$

Define

$$U_1 = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x,x_r}, \qquad U_2 = W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x,x_r}.$$

Then U_1 and U_2 are open sets, $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$, as required. ∎

Remark A topological space X is said to be *normal* if and only if, given any closed subsets F_1 and F_2 of X for which $F_1 \cap F_2 = \emptyset$, there exist open sets U_1 and U_2 of X for which $U_1 \cap U_2 = \emptyset$. Now every closed subset of a compact topological space is compact, by Lemma 4.3. It follows from Proposition 4.9 that every compact Hausdorff space is normal.

Lemma 4.10 *Let X be a compact topological space, let Y be a Hausdorff space, and let $f\colon X \to Y$ be a continuous function from X to Y. Then $f(K)$ is closed in Y for every closed set K in X.*

Proof Let K be a closed subset of X. Every closed subset of a compact topological space is compact, by Lemma 4.3. Therefore K is compact. It then follows from Lemma 4.4 that $f(K)$ is compact. But then $f(K)$ is closed, by Corollary 4.8, since Y is Hausdorff. Thus $f(K)$ is closed in Y for every closed set K in X. ∎

Theorem 4.11 *Let X be a compact topological space, let Y be a Hausdorff space, and let $f: X \to Y$ be a continuous function from X to Y which is also a bijection (i.e., f is both one-to-one and onto). Then $f: X \to Y$ is a homeomorphism.*

Proof The function f is invertible, since it is a bijection. Let $g: Y \to X$ be the inverse of $f: X \to Y$. Let U be an open set in X. Then $X \setminus U$ is closed in X, and hence $f(X \setminus U)$ is closed in Y, by Lemma 4.10. But

$$f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U).$$

It follows that $g^{-1}(U)$ is open in Y. Thus $g: Y \to X$ is continuous. We deduce that $f: X \to Y$ is a homeomorphism, as required. ∎

We recall that a function $f: X \to Y$ from a topological space X to a topological space Y is said to be an *identification map* if it is surjective and satisfies the following condition: a subset U of Y is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 4.12 *Let X be a compact topological space, let Y be a Hausdorff space. If $f: X \to Y$ is a continuous surjection then f is an identification map.*

Proof The function $f: X \to Y$ is surjective, and $f^{-1}(U)$ is open in X for any open subset U of Y, since f is continuous. Thus, to prove that $f: X \to Y$ is an identification map, it only remains to show that if U is a subset of Y such that $f^{-1}(U)$ is open in X then U is open in Y.

Let $K = X \setminus f^{-1}(U)$. If $f^{-1}(U)$ is open in X then K is closed. It follows from Lemma 4.10 that $f(K)$ is closed in Y.

Now if $x \in K$ then $x \notin f^{-1}(U)$, and hence $f(x) \in Y \setminus U$. Thus $f(K) \subset Y \setminus U$. But if y is any point of $Y \setminus U$ then $y = f(x)$ for some $x \in X$, since f is surjective, and moreover $x \in K$ (since $f(x) \notin U$). Thus $f(K) = Y \setminus U$. But $f(K)$ is closed in Y. It follows that U is open in Y, as required. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined by

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Let $q\colon [0,1] \to S^1$ be the function defined by

$$q(t) = (\cos 2\pi t, \sin 2\pi t) \qquad (t \in [0,1]).$$

The function q is surjective. Moreover the closed interval $[0,1]$ is compact, and the circle S^1 is Hausdorff. Therefore the function $q\colon [0,1] \to S^1$ is an identification map. Thus a function $f\colon S^1 \to Z$ from the circle S^1 to some topological space Z is continuous if and only if the composition function $f \circ q\colon [0,1] \to Z$ is continuous (see Lemma 2.19).

4.5 Finite Products of Compact Spaces

We shall show that a finite Cartesian product of compact spaces is compact. To prove this, we apply the following result, known as the *Tube Lemma.*

Lemma 4.13 *Let X and Y be topological spaces, let K be a compact subset of Y, and U be an open set in $X \times Y$. Let V be the subset of X defined by*

$$V = \{x \in X : \{x\} \times K \subset U\}.$$

Then V is an open set in X.

Proof Let x be a point of V. For each point y of K there exist open subsets D_y and E_y of X and Y respectively such that $(x,y) \in D_y \times E_y$ and $D_y \times E_y \subset U$. But K is compact. Therefore there exists a finite set $\{y_1, y_2, \ldots, y_k\}$ of points of K such that

$$K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k}.$$

Set

$$N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}.$$

Then N_x is an open set in X. Moreover

$$N_x \times K \subset \bigcup_{i=1}^{k} (D_{y_i} \times E_{y_i}) \subset U,$$

so that $N_x \subset V$. It follows that V is the union of the open sets N_x for all $x \in V$. Thus V is itself an open set in X, as required. ∎

Theorem 4.14 *Let X and Y be compact topological spaces. Then $X \times Y$ is compact.*

Proof Let $\mathcal{U}$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

Let x be a point of X. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, hence there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $\mathcal{U}$ such that

$$\{x\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r.$$

Let

$$V_x = \{x' \in X : \{x'\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r\}.$$

It follows from Lemma 4.13 that V_x is an open set in X. We have therefore shown that, for each point x in X, there exists an open set V_x in X containing the point x such that $V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $\mathcal{U}$.

Now $\{V_x : x \in X\}$ is an open cover of the compact space X. This cover possesses a finite subcover. Thus there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of X such that

$$X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}.$$

It follows from this that $X \times Y$ can be covered by finitely many open sets belonging to the open cover $\mathcal{U}$ (since $X \times Y$ is a finite union of sets of the form $V_x \times Y$, and each of these sets can be covered by finitely many of the open sets belonging to $\mathcal{U}$). Therefore $X \times Y$ is compact. ∎

Corollary 4.15 *Let $X_1, X_2, \ldots, X_n$ be compact topological spaces. Then the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is compact.*

Proof It follows easily from the definition of the product topology that the product topologies on $X_1 \times X_2 \times \cdots \times X_n$ and $(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$ coincide. The desired result therefore follows from Theorem 4.14 by induction on n. ∎

Theorem 4.16 *Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.*

Proof Suppose that K is compact. We show that K is closed and bounded. Note that K is closed, since $\mathbb{R}^n$ is Hausdorff, and all compact subsets of Hausdorff spaces are closed, by Corollary 4.8. Consider the open cover of $\mathbb{R}^n$ provided by the sets U_m for all positive integers m, where

$$U_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}.$$

The set K must be covered by finitely many of these open sets, since K is compact. Suppose that

$$K \subset U_{m_1} \cup U_{m_2} \cup \cdots \cup U_{m_r},$$

where $m_1 < m_2 < \cdots < m_r$. Then $K \subset U_{m_r}$, and hence $|\mathbf{x}| < m_r$ for all $\mathbf{x} \in K$. Thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 4.2), and C is the Cartesian product of n-copies of the compact set $[-L, L]$. It follows from Corollary 4.15 that C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 4.3. Thus K is compact, as required. ∎

4.6 Norms on Vector Spaces

Let V be a vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *norm* $\|.\|$ on V is a function sending an element v of V to some real number, denoted by $\|v\|$, which satisfies the following properties:—

(i) $\|v\| \geq 0$ for all $v \in V$,

(ii) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$,

(iii) $\|\lambda v\| = |\lambda| \|v\|$ for all $v \in V$ and $\lambda \in \mathbb{F}$,

(iv) $\|v\| = 0$ if and only if $v = 0$.

Example Let $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ be the functions from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\begin{aligned} \|\mathbf{x}\|_1 &= \sum_{j=1}^n |x_j|, \\ \|\mathbf{x}\|_2 &= \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}}, \\ \|\mathbf{x}\|_\infty &= \max(|x_1|, |x_2|, \ldots, |x_n|), \end{aligned}$$

for each $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ are norms on $\mathbb{R}^n$. (Note that $\|.\|_2$ is the standard Euclidean norm on $\mathbb{R}^n$, which we have denoted by $|.|$.)

A norm $\|.\|$ on a vector space V induces a corresponding distance function on V: the distance $d(v,w)$ between elements v and w of V is defined by $d(v,w) = \|v - w\|$. This distance function satisfies the metric space axioms. Thus any vector space with a norm can be regarded as a metric space. The distance function in turn induces a topology on V: a subset U of V is open (with respect to the topology induced by the norm $\|.\|$) if and only if, given any point u of U, there exists some $\delta > 0$ such that

$$\{v \in V : \|v - u\| < \delta\} \subset U.$$

Let $\|.\|_1$ and $\|.\|_2$ be norms on the vector space V. The norms $\|.\|_1$ and $\|.\|_2$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1$$

for all $v \in V$.

Lemma 4.17 *Let V be a real (or complex) vector space, and let $\|.\|_1$ and $\|.\|_2$ be norms on V. The norms $\|.\|_1$ and $\|.\|_2$ generate the same topology on V if and only if they are equivalent.*

Proof Suppose that the norms $\|.\|_1$ and $\|.\|_2$ are equivalent. Then there exist constants c and C, where $0 < c \leq C$, such that $c\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1$ for all $v \in V$. Thus, for any point u of V,

$$\begin{aligned} \{v \in V : \|v - u\|_2 < \delta\} &\subset \{v \in V : \|v - u\|_1 < \delta/c\}, \\ \{v \in V : \|v - u\|_1 < \delta\} &\subset \{v \in V : \|v - u\|_2 < C\delta\}. \end{aligned}$$

Let U be a subset of V. If U is open with respect to the topology induced by the norm $\|.\|_1$ then, given any point u of U, there exists some $\delta > 0$ such that

$$\{v \in V : \|v - u\|_1 < \delta/c\} \subset U.$$

But then

$$\{v \in V : \|v - u\|_2 < \delta\} \subset \{v \in V : \|v - u\|_1 < \delta/c\} \subset U,$$

so that U is open with respect to the topology induced by the norm $\|.\|_2$. A similar proof, using the fact that

$$\{v \in V : \|v - u\|_1 < \delta\} \subset \{v \in V : \|v - u\|_2 < C\delta\}$$

for all $\delta > 0$, shows that if U is open with respect to the topology induced by the norm $\|.\|_2$ then U is open with respect to the topology induced by the norm $\|.\|_1$. Thus equivalent norms induce the same topology on V.

Conversely suppose that $\|.\|_1$ and $\|.\|_2$ are norms on V which induce the same topology on V. Now $\{v \in V : \|v\|_2 < 1\}$ is open with respect to the topology induced by $\|.\|_2$. This set is therefore open with respect to the topology induced by $\|.\|_1$, and hence there exists some $\delta_1 > 0$ such that

$$\{v \in V : \|v\|_1 < \delta_1\} \subset \{v \in V : \|v\|_2 < 1\}.$$

A similar argument shows that there exists $\delta_2 > 0$ such that

$$\{v \in V : \|v\|_2 < \delta_2\} \subset \{v \in V : \|v\|_1 < 1\}.$$

Set $C = 2/\delta_1$ and $c = \delta_2/2$. Let v be a non-zero element of V, and let $\lambda_v = \delta_1/2\|v\|_1$ Then

$$\|\lambda_v v\|_1 = |\lambda_v|\|v\|_1 = \tfrac{1}{2}\delta_1 < \delta_1$$

and hence $\|\lambda_v v\|_2 < 1$. But

$$\|\lambda_v v\|_2 = |\lambda_v|\|v\|_2 = \frac{\|v\|_2}{C\|v\|_1}.$$

Thus $\|v\|_2 \leq C\|v\|_1$. A similar argument shows that $\|v\|_1 \leq c^{-1}\|v\|_2$. Thus $c\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1$ for all non-zero elements v of V. These inequalities also hold if $v = 0$. We conclude that if the norms $\|.\|_1$ and $\|.\|_2$ induce the same topology on V then these norms are equivalent. ∎

We shall show that any two norms on $\mathbb{R}^n$ are equivalent (Theorem 4.19). Since any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$ for all natural numbers n, this shows that any two norms on a finite-dimensional real vector space are equivalent, and thus generate the same topology on that vector space. (This result does not apply to infinite-dimensional vector spaces.)

Lemma 4.18 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then the function $\mathbf{x} \mapsto \|x\|$ is continuous with respect to the usual topology on on $\mathbb{R}^n$ (i.e., the topology on $\mathbb{R}^n$ induced by the Euclidean norm).*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Now

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$$

since

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

Also

$$\|x - y\| = \left\| \sum_{j=1}^n (x_j - y_j)\mathbf{e}_j \right\| \leq \sum_{j=1}^n |x_j - y_j| \, \|\mathbf{e}_j\| \, .$$

Let $c_j = \|\mathbf{e}_j\|$ for $j = 1, 2, \ldots, n$. It follows from Schwarz' Inequality (Lemma 1.1) that

$$\sum_{j=1}^n |x_j - y_j| \, \|\mathbf{e}_j\| = \sum_{j=1}^n |x_j - y_j| c_j \leq \left(\sum_{j=1}^n (x_j - y_j)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n c_j^2 \right)^{\frac{1}{2}} = C|\mathbf{x} - \mathbf{y}|,$$

where $C^2 = c_1^2 + c_2^2 + \cdots + c_n^2$, and where $|\mathbf{x} - \mathbf{y}|$ denotes the Euclidean norm of $\mathbf{x} - \mathbf{y}$. We conclude therefore that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq |\mathbf{x} - \mathbf{y}|$$

This shows that the function from $\mathbb{R}^n$ to $\mathbb{R}$ given by $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ with respect to the usual topology on $\mathbb{R}^n$. ∎

Theorem 4.19 *Any two norms on $\mathbb{R}^n$ are equivalent, and therefore induce the same topology on $\mathbb{R}^n$. (This topology is the usual topology on $\mathbb{R}^n$.)*

Proof Let $\|.\|_1$ be any norm on $\mathbb{R}^n$. We show that $\|.\|_1$ is equivalent to the Euclidean norm $|.|$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} = 1\}.$$

Now S^{n-1} is a compact subset of $\mathbb{R}^n$, by Theorem 4.16, since it is both closed and bounded. Also the function $f\colon S^{n-1} \to \mathbb{R}$ defined by $f(\mathbf{x}) = \|\mathbf{x}\|_1$ is continuous, by Lemma 4.18. Note that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in S^{n-1}$ (see properties (i) and (iv) in the definition of norms). It follows from Proposition 4.6 that there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in S^{n-1}$. Set $c_1 = f(\mathbf{u}) = \|\mathbf{u}\|_1$ and $C_1 = f(\mathbf{v}) = \|\mathbf{v}\|_1$. Then $0 < c_1 \leq C_1$.

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $(1/|\mathbf{x}|)\mathbf{x}$ is an element of S^{n-1}, and hence

$$c_1 \leq \left\| \frac{\mathbf{x}}{|\mathbf{x}|} \right\|_1 \leq C_1.$$

Thus $c_1|\mathbf{x}| \leq \|\mathbf{x}\|_1 \leq C_1|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|_1$ is equivalent to the Euclidean norm $|.|$ on $\mathbb{R}^n$.

If $\|.\|_2$ is any other norm on $\mathbb{R}^n$ then $\|.\|_2$ is also equivalent to the Euclidean norm on $\mathbb{R}^n$, and hence there exist constants c_2 and C_2 satisfying $0 < c_2 \leq C_2$ such that $c_2|\mathbf{x}| \leq \|\mathbf{x}\|_2 \leq C_2|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$. But then

$$\frac{c_2}{C_1}\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq \frac{C_2}{c_1}\|\mathbf{x}\|_1.$$

Thus the norms $\|.\|_1$ and $\|.\|_2$ are equivalent. This shows that any two norms on $\mathbb{R}^n$ are equivalent. It then follows from Lemma 4.17 that any two norms on $\mathbb{R}^n$ generate the same topology on $\mathbb{R}^n$. ∎

4.7 The Lebesgue Lemma

Definition Let (X, d) be a metric space, and let A be a subset of X. The *diameter* of the set A is defined to be the supremum

$$\sup_{u,v \in A} d(u, v)$$

of the distance from the point u to the point v as u and v range over all the points of the set A. (If the distance $d(u, v)$ from u to v is not bounded above as u and v range over the set A then the diameter of A is defined to be $+\infty$.)

We now state and prove the *Lebesgue Lemma*.

Lemma 4.20 (Lebesgue Lemma) *Let (X, d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \dots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \dots, r$. Let $\delta > 0$ be given by

$$\delta = \text{minimum}(\delta_1, \delta_2, \dots, \delta_r).$$

Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required. ∎

Let $\mathcal{U}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{U}$ is a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

The following result follows from the Heine-Borel Theorem (Theorem 4.2) and the Lebesgue Lemma (Lemma 4.20).

Theorem 4.21 *Let X be a topological space, and let $\mathcal{U}$ be an open cover of X. Let a and b be real numbers satisfying $a < b$, and let $\gamma\colon [a, b] \to X$ be a continuous function from the closed bounded interval $[a, b]$ into X. Then there exist $t_0, t_1, \dots, t_r \in [a, b]$, where*

$$a = t_0 < t_1 < t_2 < \cdots < t_r = b,$$

such that, for each i, $\gamma([t_{i-1}, t_i])$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.

Proof Let $\mathcal{V}$ be the open cover of $[a, b]$ consisting of all the subsets of $[a, b]$ that are of the form $\gamma^{-1}(U)$ for some open set U belonging to $\mathcal{U}$. The closed bounded interval $[a, b]$ is a compact metric space. Let $\delta > 0$ be a Lebesgue number for this open cover. Choose $t_0, t_1, \dots, t_r$ such that $t_0 = a$, $t_r = b$ and $0 < t_i - t_{i-1} < \delta$ for $i = 1, 2, \dots, r$. Then, for each i, $[t_{i-1}, t_i] \subset \gamma^{-1}(U)$ and that $\gamma([t_{i-1}, t_i] \subset U$ for some open set U belonging to the open cover $\mathcal{U}$. ∎

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f\colon X \to Y$ be a function from X to Y. The function f is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 4.22 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f\colon X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now $B_Y(y, \frac{1}{2}\varepsilon)$ is open in Y (see Lemma 1.5), and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 4.20) that there exists some $\delta > 0$ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$ which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f\colon X \to Y$ is uniformly continuous, as required. ∎

Let K be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 4.16 and Theorem 4.22 that any continuous function $f\colon K \to \mathbb{R}^k$ is uniformly continuous.