Course 121: Problems—Michaelmas Term 2003

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1. For each of the following sets, determine whether or not the set is bounded above and, if so, determine the least upper bound of the set [giving appropriate justification for your answers]:

   (i) the set $\mathbb{N}$ of all natural numbers,

   (ii) the set $\left\{ \dfrac{n-2}{n} : n \in \mathbb{N} \right\}$ consisting of all real numbers that are of the form $(n-2)/n$ for some natural number n,

   (iii) the set of all rational numbers q satisfying $q^2 < 9$.

2. Let I be a non-empty set of real numbers with the following property: if s, t and u are real numbers satisfying $s < t < u$ and if s and u both belong to I then t also belongs to I. Our objective is to classify such sets by showing that such a set I is an interval included in the list of intervals given in the lecture notes above.

   (a) Suppose that the set I is bounded above and below, so that $a = \inf I$ and $b = \sup I$ are well-defined real numbers. Let t be any real number satisfying $a < t < b$. Using the definitions of the least upper bound and the greatest lower bound, show that there exist real numbers s and u belonging to the set I which satisfy $s < t < u$. Hence show that $(a, b) \subset I$ (where $(a, b) = \{t \in \mathbb{R} : a < t < b\}$). Thus show that I is an interval of the form $[a, b]$, (a, b), $[a, b)$ or $(a, b]$.

   (b) Suppose that the set I is bounded above but is not bounded below. Let $c = \sup I$. Show that if t is any real number satisfying $t < c$ then there exist elements s and u of I satisfying $s < t < u$. Hence prove that I is an interval of the form $(-\infty, c]$ or $(-\infty, c)$.

(c) Suppose that the set I is bounded below but is not bounded above. Show that I is an interval of the form of the form $[c, +\infty)$ or $(c, +\infty)$, where $c = \inf I$.

(d) Suppose that the set I is neither bounded above nor bounded below. Prove that $I = \mathbb{R}$.

3. Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be the infinite sequences of real numbers defined by

$$a_n = \frac{15n + 7}{3n^2 - 2n}, \qquad b_n = \frac{4}{n^2} \sin n.$$

Prove that these infinite sequences are convergent, and find the limits of these sequences. [Hint: in order to prove rigorously the convergence of the sequence $b_1, b_2, b_3, \ldots$ you will probably need to make use of the formal criterion defining convergence; note that $|\sin n| \leq 1$ for all natural numbers n.]

4. Let (a_n), (b_n) and (c_n) be sequences of real numbers satisfying $a_n \leq b_n \leq c_n$ for all natural numbers n. Let l be a real number. Suppose that $a_n \to l$ and $c_n \to l$ as $n \to +\infty$. Let ε be any real number satisfying $\varepsilon > 0$. Show that there exist natural numbers N_1 and N_2 such that $b_n < l + \varepsilon$ for all n satisfying $n \geq N_1$ and $b_n > l - \varepsilon$ for all n satisfying $n \geq N_2$. Hence or otherwise prove that $b_n \to l$ as $n \to +\infty$.

5. Let a_0 be a real number satisfying $|a_0| < 1$, and let $a_0, a_1, a_2, a_3, \ldots$ be the infinite sequence of real numbers defined by $a_{n+1} = \sqrt{2 + a_n}$ for $n = 0, 1, 2, 3, \ldots$. Prove that $a_n \to 2$ as $n \to +\infty$.

6. Let (a_n) and (b_n) be infinite sequences of real numbers. Suppose that

(i) the sequence (a_n) is non-decreasing,

(ii) the sequence (b_n) is bounded above,

(iii) the sequence $(a_n - b_n)$ is convergent.

Prove that the sequences (a_n) and (b_n) are both convergent.

7. Let

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ -1 & \text{if } n \text{ is even;} \end{cases} \qquad b_n = \begin{cases} n^2 & \text{if } n \text{ is odd,} \\ -n^2 & \text{if } n \text{ is even;} \end{cases}$$

$$c_n = \begin{cases} \dfrac{n+1}{n} & \text{if } n \text{ is odd;} \\ 2n & \text{if } n \text{ is even.} \end{cases} \qquad d_n = \sin n.$$

For each of the infinite sequences (a_n), (b_n), (c_n) and (d_n), decide whether or not the sequence has a convergent subsequence.

8. Let $I_1, I_2, I_3, I_4, I_5, \ldots$ be an infinite sequence of closed intervals in $\mathbb{R}$, where each interval I_n is given by

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$$

for some real numbers a_n and b_n satisying $a_n \leq b_n$. Suppose that $I_{n+1} \subset I_n$ for each natural number n and that $b_n - a_n \to 0$ as $n \to +\infty$. Prove that there exists exactly one real number c with the property that c belongs to I_n for each natural number n.

9. Using the formal definition of limits, prove the following:—

(i) $\displaystyle\lim_{x \to 0} \sqrt{|x|} = 0$,

(ii) $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.

(iii) $\displaystyle\lim_{x \to 0} \sqrt{|x|} \sin(1/x) = 0$,

10. Let f, g and h be real-valued functions defined on some subset D of the set $\mathbb{R}$ of real numbers. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in D$, and that $f(s) = g(s) = h(s)$ for some $s \in D$. Suppose also that the functions f and h are continuous at s. Using the formal definition of continuity, prove that the function g is continuous at s.

11. Consider the situation described by Figure A. The circle passing through A, B and E is a circle of unit radius with centre O. The lines BC and AD are perpendicular to the line OA. The angle between OA and OB measured in radians is θ, where $0 < \theta < \pi/2$.

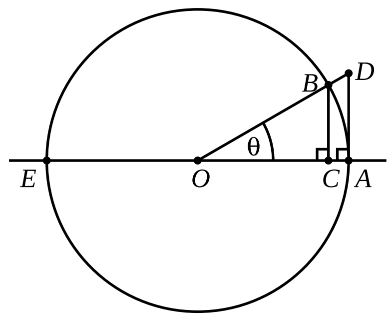

Figure A, Problem 11

(a) Write down (as a function of the angle θ) the area of the right-angled triangles OBC and ODA and the area of the sector OAB (i.e., the region bounded by the line segments OA and OB and the circular arc AB of length θ).

(b) Show that if $0 < \theta < \pi/2$ then $\sin\theta\,\cos\theta < \theta < \dfrac{\sin\theta}{\cos\theta}$.

Deduce that $\cos\theta < \dfrac{\sin\theta}{\theta} < \dfrac{1}{\cos\theta}$ when $0 < \theta < \pi/2$. Explain why these inequalities also hold when $-\pi/2 < \theta < 0$.

(c) Using part (b), or otherwise, show that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

[Note: you may assume that the sine and cosine functions are continuous.]

(d) Explain why $\dfrac{1-\cos\theta}{\sin\theta} = \tan\frac{1}{2}\theta = \dfrac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}$.

Use this result to show that $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = 0$.

Using this result, and part (c), show that $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$.

12. Determine whether or not the functions g and h are continuous at $x = 0$, where

$$g(x) = \begin{cases} \dfrac{2}{1+x^2}\cos x \sin\left(\dfrac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and $h(x) = x^2 g(x)$ for all $x \in \mathbb{R}$.

13. Let D be a subset of $\mathbb{R}$ and let $f\colon D \to \mathbb{R}$ be a function mapping D into $\mathbb{R}$. Let $(a_n : n \in \mathbb{N})$ be a sequence of real numbers belonging to D. Suppose that $a_n \to s$ as $n \to +\infty$, where s is some limit point of D, and that $f(x) \to l$ as $x \to s$ in D. Suppose also that the number of values of n for which $a_n = s$ is finite. Prove that $f(a_n) \to l$ as $n \to +\infty$.

14. Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function defined on the set $\mathbb{R} \setminus \{0\}$ of all non-zero real numbers. Suppose that $f(x) \geq f(y)$ for all non-zero real numbers x and y satisfying $|x| \leq |y|$. Suppose also that the function f is bounded above on $\mathbb{R} \setminus \{0\}$ (i.e., there exists a constant B such that $f(x) \leq B$ for all non-zero real numbers x). Prove that $\lim_{x \to 0} f(x)$ exists. [Hint: consider least upper bounds.]