Course 121: Problems—Hilary Term 2004

D. R. Wilkins

1. The sine and cosine functions satisfy

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

Using these results, together with the addition formulae for the sine and cosine functions, prove that

$$\frac{d}{dx} \sin(x) = \cos x, \qquad \frac{d}{dx} \cos(x) = -\sin x.$$

2. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be a 3-times differentiable function on $\mathbb{R}$. Let a and b be real numbers satisfying $a < b$. Suppose that $f(a) = 0$, $f(b) = 0$, $f'(a) = 0$ and $f'(b) = 0$. prove that there exists some s in the range $a < s < b$ for which $f'''(s) = 0$.

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be a 5-times differentiable function. Let a, b and c be real numbers satisfying $a < b < c$. Suppose that

$$f(a) = f'(a) = f(b) = f'(b) = f(c) = f'(c) = 0.$$

Prove that there exists some s satisfying $a < s < c$ for which $f^{(5)}(s) = 0$.

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$ which is $2k + 1$ times differentiable, for some non-negative integer k. Let a and b be real numbers satisfying $a < b$. Suppose that $f^{(j)}(a) = 0$ and $f^{(j)}(b) = 0$ for $j = 0, 1, \ldots, k$. Prove that there exists some $\xi \in \mathbb{R}$ satisfying $a < \xi < b$ for which $f^{(2k+1)}(\xi) = 0$.

3. (a) Using the Intermediate Value Theorem and Rolle's Theorem, show that the polynomial $x^5 + 2x^3 + 7x - 13$ has exactly one real root.

(b) Prove that the polynomial $x^4 + x^2 - 7x - 2$ has exactly 2 distinct real roots, where one of these roots is positive and the other is negative.

4. (a) Let $f: I \to \mathbb{R}$ be a differentiable function defined on some open interval I. Suppose that there exists some non-negative real number K such that $|f'(x)| \leq K$ for all $x \in I$. Prove that $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$ for all $x_1, x_2 \in I$.

   (b) Show that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.

5. Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose that $f(0) = a$, $f'(0) = b$ and $f''(x) \geq -c$ for all $x > 0$, where $c > 0$. Prove that $f(x) > a + bx - cx^2$ for all $x > 0$.

6. Prove that $x - x^3 \leq \sin x \leq x$ for all $x \geq 0$.

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) \geq 0$ for all $x \in [a, b]$, where a and b are real numbers satisfying $a < b$. Suppose also that the derivative f' of f is continuous and that $f'(x) > 0$ for at least one value of x in the interval (a, b). Prove that $f(b) > f(a)$.

8. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f(0) = 0$ and $|f'(x)| \leq A|x|^n$ for some $A \geq 0$ and some non-negative integer n. Use the Cauchy Mean Value Theorem (with an appropriate choice of the function g occurring in the statement of that theorem) to show that $|f(x)| \leq \dfrac{A}{n+1}|x|^{n+1}$ for all $x \in \mathbb{R}$.

   (b) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos\left(\dfrac{\pi}{2}\cos x\right)$. Show that $|f(x)| \leq \frac{\pi}{4}|x|^2$ for all x.

9. Evaluate the following limits, using l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin\sin x}{\sin x}, \qquad \lim_{x\to 0} \frac{\sin\sin\sin x}{x}, \qquad \lim_{x\to 2} \frac{x^3 - x^2 - 8x + 12}{x^3 - 3x^2 + 4},$$

$$\lim_{x\to 5} \frac{x^3 - 12x^2 + 45x - 50}{x^3 - 9x^2 + 15x + 25}, \qquad \lim_{x\to 0} \frac{1 - \cos x}{\sin^2 x}, \qquad \lim_{x\to 0} \frac{\cos(x^2) - 1}{\sin x^4}.$$

10. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathbb{R}$, where $g(x)$ and $g'(x)$ are non-zero for all sufficiently large x. Suppose that

$f(x) \to 0$ and $g(x) \to 0$ as $x \to +\infty$ and that $\lim_{x\to+\infty} \dfrac{f'(x)}{g'(x)}$ exists. Prove that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}.$$

[Hint: consider the limits of $F(u)/G(u)$ and $F'(u)/G'(u)$ as $u \to 0$ from above, where $F(u) = f(1/u)$ and $G(u) = g(1/u)$.]

11. The exponential function exp satisfies $\dfrac{d}{dx}\exp(x) = \exp(x)$ for all x (where $\exp(x) = e^x$). Use Taylor's Theorem to prove that

$$\exp(x) = \lim_{m\to+\infty} \sum_{n=0}^{m} \frac{x^n}{n!}$$

for all real numbers x. [You might need to consider separately the cases $x > 0$ and $x < 0$.]

12. Let $f(x) = x^2$. The purpose of this question is to show from first principles that the function f is Riemann-integrable on $[0, s]$, where $s > 0$, and to evaluate the Riemann integral of f on this interval.

(a) For each natural number n let P_n denote the partition $\{x_0, x_1, \ldots, x_n\}$ of $[0, s]$ into n subintervals of equal length, given by $x_i = is/n$ for $i = 0, 1, \ldots, n$. By making use of the identities

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1), \qquad \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1),$$

or otherwise, show that the lower sum $L(P_n, f)$ is given by

$$L(P_n, f) = \frac{s^3}{3}\left(1 - \frac{3}{2n} + \frac{1}{2n^2}\right)$$

and calculate the upper sum $U(P_n, f)$.

(b) Show that $\lim_{n\to+\infty} L(P_n, f) = \frac{1}{3}s^3$ and $\lim_{n\to+\infty} U(P_n, f) = \frac{1}{3}s^3$. Hence prove that the function is Riemann-integrable, and $\displaystyle\int_0^s x^2\,dx = \frac{1}{3}s^3$.

13. Let $f(x) = e^{kx}$, where $k \geq 0$. The purpose of this question is to show from first principles that the function f is Riemann-integrable on $[0, s]$, where $s > 0$, and to evaluate the Riemann integral of f on this interval.

(a) For each natural number n let P_n denote the partition $\{x_0, x_1, \ldots, x_n\}$ of $[0, s]$ into n subintervals of equal length, given by $x_i = is/n$ for $i = 0, 1, \ldots, n$). By making use of the identities

$$1 + u + u^2 + \cdots + u^{n-1} = \frac{u^n - 1}{u - 1} \quad (u \neq 1), \qquad \lim_{h \to 0} \frac{1}{h}(e^h - 1) = 1,$$

or otherwise, show that the lower sum $L(P_n, f)$ is given by

$$L(P_n, f) = \frac{s\left(e^{ks} - 1\right)}{n\left(e^{\frac{ks}{n}} - 1\right)},$$

and calculate the upper sum $U(P_n, f)$.

(b) Show that $\lim_{n \to +\infty} L(P_n, f) = \lim_{n \to +\infty} U(P_n, f) = \dfrac{1}{k}(e^{ks} - 1)$. Hence prove that the function is Riemann-integrable on $[0, s]$, and

$$\int_0^s e^{kx}\, dx = \frac{1}{k}(e^{ks} - 1).$$

14. Let $g\colon [0, 1] \to \mathbb{R}$ be the function on $[0, 1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \neq \frac{1}{2}; \\ 0 & \text{if } x = \frac{1}{2}. \end{cases}$$

Given δ satisfying $0 < \delta < \frac{1}{2}$, calculate the upper sum $U(g, Q_\delta)$ and the lower sum $U(g, Q_\delta)$ for the partition Q_δ of $[0, 1]$, where $Q_\delta = \left\{0, \frac{1}{2} - \delta, \frac{1}{2} + \delta, 1\right\}$. Calculate $\lim_{\delta \to 0} U(g, Q_\delta)$ and $\lim_{\delta \to 0} L(g, Q_\delta)$. Explain why the function g is Riemann-integrable on $[0, 1]$ and write down the value of the Riemann integral of g on $[0, 1]$.

15. (a) Prove that if a and b are real numbers satisfying $a < b$ and if $f\colon [a, b] \to \mathbb{R}$ is a continuous real-valued function defined on the closed interval $[a, b]$ then

$$\frac{d}{dx} \int_x^b f(t)\, dt = -f(x)$$

for all $x \in (a, b)$.

(b) Evaluate $\displaystyle \frac{d}{dx} \int_{\cos x - 5}^{\sin x + 2} t^5 e^{-t}\, dt.$

16. Let a and h be real numbers, and let f be a real-valued function, defined on some open interval containing a and $a+h$, with the property that the first k derivatives $f', f'', \dots, f^{(k)}$ of f exist and are continuous on this interval.

(a) Let

$$r_m(a,h) = \frac{h^m}{(m-1)!} \int_0^1 (1-x)^{m-1} f^{(m)}(a+xh)\,dx$$

for $m = 1, 2, \dots, k-1$. Show that

$$r_m(a,h) = \frac{h^m}{m!} f^{(m)}(a) + r_{m+1}(a,h).$$

(b) Using (a) and induction on k, show that

$$f(a+h) = f(a) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(a) + \frac{h^k}{(k-1)!} \int_0^1 (1-x)^{k-1} f^{(k)}(a+xh)\,dx.$$

(c) By applying (b) to the function $t \mapsto x^\alpha$, show that

$$(1+h)^\alpha = 1 + \sum_{n=1}^{k-1} C_{n,\alpha} h^n + R_k(h)$$

for all $\alpha \in \mathbb{R}$ and $h > -1$, where

$$\begin{aligned} C_{n,\alpha} &= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \\ R_k(h) &= k C_{k,\alpha} h^k \int_0^1 (1-x)^{k-1} (1+xh)^{\alpha-k}\,dx. \end{aligned}$$

Using the fact that

$$0 \le \frac{1-t}{1+th} \le 1$$

for all t and h satisfying $0 \le t \le 1$ and $h > -1$, show that

$$|R_k(h)| \le k|C_{k,\alpha}||h|^{k-1}|I_\alpha(h)|,$$

for all $h > -1$, where

$$I_\alpha(h) = h \int_0^1 (1+xh)^{\alpha-1}\,dx = \begin{cases} \alpha^{-1}\left((1+h)^\alpha - 1\right) & \text{if } \alpha \neq 0; \\ \log(1+h) & \text{if } \alpha = 0. \end{cases}$$

(Note that the value of $I_\alpha(h)$ is independent of k.) Hence prove that $|R_k(h)| \to 0$ as $k \to +\infty$ for all h satisfying $|h| < 1$, and thus

$$(1+h)^\alpha = 1 + \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} h^n$$

whenever $|h| < 1$.

17. (a) Let $h_n(x) = \dfrac{n}{x^2 + n^2}$ for all natural numbers n and real numbers t. Prove that the sequence $h_1, h_2, h_3, \ldots$ of functions converges uniformly on $\mathbb{R}$ to the zero function.

(b) Calculate $\int_{-\infty}^{+\infty} h_n(x)\, dx$ for all natural numbers n. Is it true that

$$\lim_{n\to+\infty} \int_{-\infty}^{+\infty} h_n(x)\, dx = \int_{-\infty}^{+\infty} \left(\lim_{n\to+\infty} h_n(x) \right) dx$$

in this case?