Course 121: Trinity Term 2004

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7 Infinite Series

An infinite series is the formal sum of the form $a_1 + a_2 + a_3 + \cdots$, where each number a_n is real or complex. Such a formal sum is also denoted by $\sum_{n=1}^{+\infty} a_n$.

Sometimes it is appropriate to consider infinite series $\sum_{n=m}^{+\infty} a_n$ of the form $a_m + a_{m+1} + a_{m+2} + \cdots$, where $m \in \mathbb{Z}$. Clearly results for such sequences may be deduced immediately from corresponding results in the case $m = 1$.

Definition An infinite series $\sum_{n=1}^{+\infty} a_n$ is said to *converge* to some complex number s if and only if, given any $\varepsilon > 0$, there exists some natural number N such that $\left| \sum_{n=1}^{m} a_n - s \right| < \varepsilon$ for all natural numbers m satisfying $m \geq N$. If the infinite series $\sum_{n=1}^{+\infty} a_n$ converges to s then we write $\sum_{n=1}^{+\infty} a_n = s$. An infinite series is said to be *divergent* if it is not convergent.

For each natural number m, the *mth partial sum* s_n of the infinite series $\sum_{n=1}^{+\infty} a_n$ is given by $s_m = a_1 + a_2 + \cdots + a_m$. Note that $\sum_{n=1}^{+\infty} a_n$ converges to some complex number s if and only if $s_m \to s$ as $m \to +\infty$. The following proposition therefore follows immediately on applying the results of Proposition 6.3.

Proposition 7.1 *Let* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *be convergent infinite series. Then* $\sum_{n=1}^{+\infty} (a_n+b_n)$ *is convergent, and* $\sum_{n=1}^{+\infty} (a_n+b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n$. *Also* $\sum_{n=1}^{+\infty} (\lambda a_n) =$

$\lambda \sum_{n=1}^{+\infty} a_n$ *for any complex number* λ.

If $\sum_{n=1}^{+\infty} a_n$ is convergent then $a_n \to 0$ as $n \to +\infty$. Indeed

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} (s_n - s_{n-1}) = \lim_{n\to+\infty} s_n - \lim_{n\to+\infty} s_{n-1} = s - s = 0,$$

where $s_m = \sum_{n=1}^{m} a_n$ and $s = \sum_{n=1}^{+\infty} a_n = \lim_{m\to+\infty} s_m$. However the condition that $a_n \to 0$ as $n \to +\infty$ is not in itself sufficient to ensure convergence. For example, the series $\sum_{n=1}^{+\infty} 1/n$ will be shown to be divergent.

Proposition 7.2 *Let $a_1, a_2, a_3, a_4, \ldots$ be an infinite sequence of real numbers. Suppose that $a_n \geq 0$ for all n. Then $\sum_{n=1}^{+\infty} a_n$ is convergent if and only if there exists some real number C such that $a_1 + a_2 + \cdots + a_n \leq C$ for all n.*

Proof The sequence $s_1, s_2, s_3, \ldots$ of partial sums of the series $\sum_{n=1}^{+\infty} a_n$ is non-decreasing, since $a_n \geq 0$ for all n. The result therefore is a consequence of the fact that a non-decreasing sequence of real numbers is convergent if and only if it is bounded above (see Theorem 2.3). ∎

7.1 Some Important Examples

Example Let z be a complex number. The infinite series $1 + z + z^2 + z^3 + \cdots$ is referred to as the *geometric series*. This series is clearly divergent whenever $|z| \geq 1$, since z^n does not converge to 0 as $n \to +\infty$. We claim that the series converges to $1/(1-z)$ whenever $|z| < 1$. Now

$$1 + z + z^2 + \ldots z^m = \frac{1 - z^{m+1}}{1 - z}.$$

(To see this, multiply both sides of the equation by $1 - z$.) Thus if $|z| < 1$ then

$$\lim_{m\to+\infty} (1 + z + z^2 + \ldots z^m) = \frac{1}{1-z}.$$

It is important that the above identity giving the sum of the geometric series is used only in situations in which the series is convergent. Nonsensical results will be obtained if one tries to apply the result in cases where the series diverges. Indeed if one were to set $z = +2$ one would obtain the nonsensical identity "$1 + 2 + 4 + 8 + 16 + 32 + \cdots = -1$"

Example We show that the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent. Let s_m denote the mth partial sum of this series, given by $s_m = \sum_{n=1}^{m} \frac{1}{n}$. We claim that $s_{2^k} \geq (k+2)/2$ for all natural numbers k. The result is clearly valid when $k = 1$. Now Suppose that $s_{2^{k-1}} \geq k/2$. Then

$$\begin{aligned} s_{2^k} &= s_{2^{k-1}} + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} \\ &\geq s_{2^{k-1}} + 2^{k-1} \times \frac{1}{2^k} = s_{2^{k-1}} + \frac{1}{2}. \end{aligned}$$

It therefore follows by induction on k that $s_{2^k} \geq (k+2)/2$ for all natural numbers k. Thus the sequence $s_1, s_2, s_3, \ldots$ is not bounded above, and so cannot converge. We conclude therefore that the infinite series $\sum_{n=1}^{+\infty} 1/n$ is divergent.

Example Let α be a real number satisfying $\alpha > 1$. We show that $\sum_{n=1}^{+\infty} 1/n^\alpha$ is convergent. Let s_m denote the mth partial sum of this series, given by $s_m = \sum_{n=1}^{m} \frac{1}{n^\alpha}$. Then

$$\begin{aligned} s_{2^k} &= 1 + \frac{1}{2^\alpha} + \left(\frac{1}{3^\alpha} + \frac{1}{4^\alpha}\right) + \left(\frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha} + \frac{1}{8^\alpha}\right) \\ &\quad + \cdots + \left(\frac{1}{(2^{k-1}+1)^\alpha} + \frac{1}{(2^{k-1}+2)^\alpha} + \cdots + \frac{1}{2^{k\alpha}}\right) \\ &< 1 + 1 + 2 \times \frac{1}{2^\alpha} + 4 \times \frac{1}{4^\alpha} + \cdots + 2^{(k-1)} \times \frac{1}{2^{(k-1)\alpha}} \\ &< C \end{aligned}$$

for all natural numbers k, where

$$C = 1 + \sum_{n=0}^{+\infty} \frac{1}{2^{(\alpha-1)n}} = 1 + \frac{1}{1 - 2^{1-\alpha}}.$$

But the sequence $s_1, s_2, s_3, \ldots$ of partial sums of the series is increasing. Therefore $S_m < C$ for all m (provided that $\alpha > 1$). It follows immediately from Proposition 7.2 that $\sum_{n=1}^{+\infty} 1/n^\alpha$ converges when $\alpha > 1$.

7.2 The Comparison Test and Ratio Test

Proposition 7.3 *An infinite series* $\sum_{n=1}^{+\infty} a_n$ *of real or complex numbers is convergent if and only if, given any* $\varepsilon > 0$*, there exists some natural number* N *with the property that*

$$|a_m + a_{m+1} + \cdots + a_{m+k}| < \varepsilon$$

for all m *and* k *satisfying* $m \geq N$ *and* $k \geq 0$.

Proof The stated criterion is equivalent to the condition that the sequence of partial sums of the series be a Cauchy sequence. The required result thus follows immediately from Cauchy's Criterion for convergence (Theorem 6.6). ∎

Proposition 7.4 (Comparison Test) *Suppose that* $0 \leq |a_n| \leq b_n$ *for all* n*, where* a_n *is complex,* b_n *is real, and* $\sum_{n=1}^{+\infty} b_n$ *is convergent. Then* $\sum_{n=1}^{+\infty} a_n$ *is convergent.*

Proof Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $b_m + b_{m+1} + \cdots + b_{m+k} < \varepsilon$ for all m and k satisfying $m \geq N$ and $k \geq 0$. But then

$$\begin{aligned} |a_m + a_{m+1} + \cdots + a_{m+k}| &\leq |a_m| + |a_{m+1}| + \cdots + |a_{m+k}| \\ &\leq b_m + b_{m+1} + \cdots + b_{m+k} < \varepsilon \end{aligned}$$

when $m \geq N$ and $k \geq 0$. Thus $\sum_{n=1}^{+\infty} a_n$ is convergent, by Proposition 7.3. ∎

Let us apply the Comparison Test in the case when a_n and b_n are non-negative real numbers satisfying $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{+\infty} b_n$ is convergent, then so is $\sum_{n=1}^{+\infty} a_n$. Thus if $\sum_{n=1}^{+\infty} a_n$ is divergent then so is $\sum_{n=1}^{+\infty} b_n$. These results also follow directly from Proposition 7.2.

Example The series $\sum_{n=1}^{+\infty} 1/n^\alpha$ diverges for all $\alpha \leq 1$, since $1/n \leq 1/n^\alpha$ for all n and $\sum_{n=1}^{+\infty} 1/n$ is divergent.

Example Comparison with the geometric series shows that the infinite series $\sum_{n=1}^{+\infty} z^n/n$ is convergent whenever $|z| < 1$.

Example The infinite series $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin n}{10n^2 - 7n + 13}$ is convergent. Indeed

$$\left|\frac{\sin n}{10n^2 - 7n + 13}\right| \leq \frac{1}{3n^2}$$

for all n, and $\sum_{n=1}^{+\infty} 1/(3n^2)$ is convergent.

Proposition 7.5 (Ratio Test) *Let $a_1, a_2, a_3 \ldots$ be complex numbers. Suppose that $r = \displaystyle\lim_{n\to+\infty} \frac{a_{n+1}}{a_n}$ exists and satisfies $|r| < 1$. Then $\sum_{n=1}^{+\infty} a_n$ is convergent.*

Proof Choose ρ satisfying $|r| < \rho < 1$. Then there exists some natural number N such that $|a_{n+1}/a_n| < \rho$ for all $n \geq N$. Let

$$K = \text{maximum}\left(\frac{|a_1|}{\rho}, \frac{|a_2|}{\rho^2}, \frac{|a_3|}{\rho^3}, \ldots \frac{|a_N|}{\rho^N}\right).$$

Now $|a_{n+1}| \leq \rho|a_n|$ whenever $n \geq N$. Therefore $|a_n| \leq \rho^{n-N}|a_N| \leq K\rho^n$ whenever $n \geq N$. But the choice of K also ensures that $|a_n| \leq K\rho^n$ when $n < N$. Moreover $\sum_{n=1}^{+\infty} K\rho^n$ converges, since $\rho < 1$. The desired result therefore follows on applying the Comparison Test (Proposition 7.4). ∎

Example Let z be a complex number. Then $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n!}$ converges for all values of z. For if $a_n = z^n/n!$ then $a_{n+1}/a_n = z/(n+1)$, and hence $a_{n+1}/a_n \to 0$ as $n \to +\infty$. The result therefore follows on applying the Ratio Test.

Let $\sum_{n=1}^{+\infty} a_n$ be an infinite series for which $r = \lim_{n\to+\infty} a_{n+1}/a_n$ is well-defined. The series clearly diverges if $|r| > 1$, since $|a_n|$ increases without limit as $n \to +\infty$. If however $|r| = 1$ then the Ratio Test is of no help in deciding whether or not the series converges, and one must try other more sensitive tests.

7.3 Convergence of Alternating Series

Ans *alternating series* is an infinite series with the property that the signs of the summands are alternately positive and negative.

Theorem 7.6 (Alternating Series Test) *Let $a_1, a_2, a_3, \ldots$ be non-negative real numbers. Suppose that $a_1 \geq a_2 \geq a_3 \geq \cdots$ and that $a_n \to 0$ as $n \to +\infty$. Then the infinite series $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n$ is convergent.*

Proof For each natural number m let $s_m = \sum_{n=1}^{m} (-1)^{n-1} a_n$. Now

$$s_{2k+1} = s_{2k-1} - a_{2k} + a_{2k+1} \leq s_{2k-1}, \qquad s_{2k+2} = s_{2k} + a_{2k+1} - a_{2k+2} \geq s_{2k}$$

for all natural numbers k (since $a_{2k} \geq a_{2k+1} \geq a_{2k+2}$). Therefore the subsequence $s_1, s_3, s_5, s_7, \ldots$ is non-increasing and the subsequence $s_2, s_4, s_6, s_8, \ldots$ is non-decreasing. But $s_2 \leq s_{2k} \leq s_{2k-1} \leq s_1$ for all natural numbers k. Thus these subsequences are bounded, and are therefore convergent (since any bounded non-increasing or non-decreasing sequence of real numbers is convergent, by Theorem 2.3). Moreover the subsequences have the same limit, since

$$\lim_{k \to +\infty} s_{2k} - \lim_{k \to +\infty} s_{2k-1} = \lim_{k \to +\infty} (s_{2k} - s_{2k-1}) = - \lim_{k \to +\infty} a_{2k} = 0.$$

We claim that $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n = s$, where $s = \lim_{k \to +\infty} s_{2k} = \lim_{k \to +\infty} s_{2k-1}$.

Let $\varepsilon > 0$ be given. Then there exist natural numbers K_1 and K_2 such that $|s - s_{2k-1}| < \varepsilon$ whenever $k \geq K_1$ and $|s - s_{2k}| < \varepsilon$ whenever $k \geq K_2$. Choose N such that $N \geq 2K_1 - 1$ and $N \geq 2K_2$. Then $|s - s_m| < \varepsilon$ whenever $m \geq N$. Thus $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n = \lim_{m \to +\infty} s_m = s$, as required. ∎

Example The infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, by the Alternating Series Test (Theorem 7.6).

7.4 Absolute Convergence

Definition An infinite series $\sum_{n=1}^{+\infty} a_n$ is said to be *absolutely convergent* if the infinite series $\sum_{n=1}^{+\infty} |a_n|$ is convergent. (A convergent series which is not absolutely convergent is said to be *conditionally convergent.*)

An absolutely convergent infinite series is convergent, and the sum of any two absolutely convergent series is itself absolutely convergent. These results follow on applying the Comparison Test (Proposition 7.4). Moreover the following criterion for absolute convergence follows directly from Proposition 7.3.

Proposition 7.7 *An infinite series* $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent if and only if, given any* $\varepsilon > 0$, *there exists some natural number* N *such that*

$$|a_m| + |a_{m+1}| + \cdots + |a_{m+k}| < \varepsilon$$

for all m *and* k *satisfying* $m \geq N$ *and* $k \geq 0$.

Many of the tests for convergence described above do in fact test for absolute convergence; these include the Comparison Test and the Ratio Test.

A *rearrangement* of a given infinite series is a new infinite series obtained on summing up the terms of the given series in a different order. Thus for example

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

is a rearrangement of the infinite series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots.$$

It might be supposed that, if one rearranges a convergent infinite series, then the rearranged series is also convergent and has the same sum as the original series. This result does in fact hold for *absolutely convergent* series (see Theorem 7 in Chapter 22 of *Calculus*, by M. Spivak). If however an infinite series is not absolutely convergent, then, given any real number α, there always exists a rearrangement of the series converging to α (see Theorem 6 in Chapter 22 of *Calculus*, by M. Spivak). Thus particular care must be exercised whenever the order of the terms of an infinite series is changed.

7.5 The Cauchy Product of Infinite Series

The *Cauchy product* of two infinite series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ is defined to be the series $\sum_{n=0}^{+\infty} c_n$, where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0.$$

The convergence of $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ is not in itself sufficient to ensure the convergence of the Cauchy product of these series. Convergence is however assured provided that the series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ are absolutely convergent.

Theorem 7.8 *The Cauchy product $\sum_{n=0}^{+\infty} c_n$ of two absolutely convergent infinite series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ is absolutely convergent, and*

$$\sum_{n=0}^{+\infty} c_n = \left(\sum_{n=0}^{+\infty} a_n\right)\left(\sum_{n=0}^{+\infty} b_n\right).$$

Proof For each non-negative integer m, let

$$\begin{aligned} S_m &= \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq j \leq m, \quad 0 \leq k \leq m\}, \\ T_m &= \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : j \geq 0, \ k \geq 0, \ 0 \leq j + k \leq m\}. \end{aligned}$$

Now $\sum_{n=0}^{m} c_n = \sum_{(j,k)\in T_m} a_j b_k$ and $\left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right) = \sum_{(j,k)\in S_m} a_j b_k$. Also

$$\sum_{n=0}^{m} |c_n| \leq \sum_{(j,k)\in T_m} |a_j||b_k| \leq \sum_{(j,k)\in S_m} |a_j||b_k| \leq \left(\sum_{n=0}^{+\infty} |a_n|\right)\left(\sum_{n=0}^{+\infty} |b_n|\right),$$

since $|c_n| \leq \sum_{j=0}^{n} |a_j||b_{n-j}|$ and the infinite series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ are absolutely convergent. It follows from Proposition 7.2 that the Cauchy product $\sum_{n=0}^{+\infty} c_n$ is absolutely convergent, and is thus convergent. Moreover

$$\begin{aligned} &\left|\sum_{n=0}^{2m} c_n - \left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right)\right| \\ &\qquad = \left|\sum_{(j,k)\in T_{2m}\setminus S_m} a_j b_k\right| \\ &\qquad \leq \sum_{(j,k)\in T_{2m}\setminus S_m} |a_j b_k| \leq \sum_{(j,k)\in S_{2m}\setminus S_m} |a_j b_k| \\ &\qquad = \left(\sum_{n=0}^{2m} |a_n|\right)\left(\sum_{n=0}^{2m} |b_n|\right) - \left(\sum_{n=0}^{m} |a_n|\right)\left(\sum_{n=0}^{m} |b_n|\right), \end{aligned}$$

since $S_m \subset T_{2m} \subset S_{2m}$. But

$$\begin{aligned}\lim_{m\to+\infty}\left(\sum_{n=0}^{2m}|a_n|\right)\left(\sum_{n=0}^{2m}|b_n|\right) &= \left(\sum_{n=0}^{+\infty}|a_n|\right)\left(\sum_{n=0}^{+\infty}|b_n|\right) \\ &= \lim_{m\to+\infty}\left(\sum_{n=0}^{m}|a_n|\right)\left(\sum_{n=0}^{m}|b_n|\right),\end{aligned}$$

since the infinite series $\sum\limits_{n=0}^{+\infty} a_n$ and $\sum\limits_{n=0}^{+\infty} b_n$ are absolutely convergent. It follows that

$$\lim_{m\to+\infty}\left(\sum_{n=0}^{2m} c_n - \left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right)\right) = 0,$$

and hence

$$\sum_{n=0}^{+\infty} c_n = \lim_{m\to+\infty}\sum_{n=0}^{2m} c_n = \left(\sum_{n=0}^{+\infty} a_n\right)\left(\sum_{n=0}^{+\infty} b_n\right),$$

as required. ∎

Example It follows from Theorem 7.8 that $\exp(z+w) = \exp(z)\exp(w)$ for all complex numbers z and w, where $\exp(z) = \sum\limits_{n=0}^{+\infty}\dfrac{z^n}{n!}$. Indeed the infinite series defining $\exp(z+w)$ is the Cauchy product of the infinite series defining $\exp(z)$ and $\exp(w)$, since

$$\sum_{n=0}^{m}\frac{z^n}{n!}\frac{w^{m-n}}{(m-n)!} = \frac{1}{m!}\sum_{n=0}^{m}\binom{m}{n}z^n w^{m-n} = \frac{1}{m!}(z+w)^m$$

by the Binomial Theorem, where $\dbinom{m}{n} = \dfrac{m!}{n!(m-n)!}$.

7.6 Uniform Convergence for Infinite Series

Let $f_1, f_2, f_2, \ldots$ be complex-valued functions defined over a subset D of $\mathbb{C}$. The infinite series series $\sum\limits_{n=1}^{+\infty} f_n(z)$ is said to converge *uniformly* on D to some function s if, given any $\varepsilon > 0$, there exists some natural number N (which does not depend on the value of z) such that $\left|s(z) - \sum\limits_{m=0}^{n} f_m(z)\right| < \varepsilon$ whenever $z \in D$ and $n \geq N$.

Note that an infinite series $\sum_{n=0}^{+\infty} f_n(z)$ of functions converges uniformly if and only if the partial sums of this series converge uniformly. It follows immediately from Theorem 6.12 that if the functions f_n are continuous on D, and if the series $\sum_{n=0}^{+\infty} f_n(z)$ converges uniformly on D to some function, then that function is also continuous on D.

Proposition 7.9 (The Weierstrass M-Test) *Let D be a subset of $\mathbb{C}$ and let $f_1, f_2, f_3, \ldots$ be a sequence of functions from D to $\mathbb{C}$, let $M_1, M_2, M_3, \ldots$ be non-negative real numbers satisfying $|f_n(z)| \leq M_n$ for all natural numbers n and $z \in D$. Suppose that $\sum_{n=1}^{+\infty} M_n$ converges. Then $\sum_{n=1}^{+\infty} f_n(z)$ converges absolutely and uniformly on D.*

Proof It follows immediately from the Comparison Test (Proposition 7.4) that the series $\sum_{n=1}^{+\infty} f_n(z)$ is absolutely convergent for all $z \in D$. We must show that the convergence is uniform.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $\sum_{n=N}^{+\infty} M_n < \frac{1}{2}\varepsilon$, since $\sum_{n=1}^{+\infty} M_n$ converges. Now if m and k are integers satisfying $m \geq N$ and $k \geq 0$ then

$$\left| \sum_{n=1}^{m+k} f_n(z) - \sum_{n=1}^{m} f_n(z) \right| = \left| \sum_{n=m+1}^{m+k} f_n(z) \right| \leq \sum_{n=m+1}^{m+k} M_m \leq \sum_{n=N}^{+\infty} M_n < \tfrac{1}{2}\varepsilon$$

for any $z \in D$. On taking the limit as $k \to +\infty$, we see that

$$\left| \sum_{n=1}^{+\infty} f_n(z) - \sum_{n=1}^{m} f_n(z) \right| \leq \tfrac{1}{2}\varepsilon < \varepsilon$$

for all $z \in D$ and $m \geq N$. However N has been chosen independently of z. Thus the infinite series converges uniformly on D, as required. ∎

7.7 Power Series

A *power series* is an infinite series of the form $\sum_{n=0}^{+\infty} a_n(z - z_0)^n$, where the coefficients $a_0, a_1, a_2, \ldots$ are complex numbers.

Definition Let $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ be a power series centred on some complex number z_0. Suppose that the set of complex numbers z for which the power series converges is bounded. Then the *radius of convergence* R_0 of the power series is defined to be the smallest non-negative real number with the property that every complex number z for which the power series converges satisfies $|z-z_0| \leq R_0$. The circle $\{z \in \mathbb{C} : |z-z_0| = R_0\}$ is then referred to as the *circle of convergence* of the power series. We set $R_0 = +\infty$ if the set of complex numbers z for which the power series converges is unbounded.

Theorem 7.10 *Let $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ be a power series with radius of convergence R_0, and let $s(z)$ denote the sum of the power series at those complex numbers z at which the series converges.*

(i) *If $R_0 = +\infty$ then $s(z)$ is a continuous function of z defined over the entire complex plane $\mathbb{C}$.*

(ii) *If $R_0 < +\infty$ then $s(z)$ is a continuous function of z defined over the whole of the disk*
$$\{z \in \mathbb{C} : |z-z_0| < R_0\}$$
bounded by the circle of convergence of the power series.

Proof Let z_1 be any complex number satisfying $|z_1 - z_0| < R_0$. Then we can choose R such that $|z_1 - z_0| < R < R_0$ and $R < +\infty$. Now it follows from the definition of the radius of convergence that there exists some complex number w such that $R < |w| < R_0$ and $\sum_{n=0}^{+\infty} a_n w^n$ converges. Choose some positive real number A with the property that $|a_n w^n| \leq A$ for all n, and set $\rho = R/|w|$ and $M_n = A\rho^n$. If $|z-z_0| < R$ then $|a_n(z-z_0)^n| \leq |a_n|R^n \leq A\rho^n = M_n$ for all n. Also $\sum_{n=0}^{+\infty} M_n$ converges to $A/(1-\rho)$. Thus we can apply the Weierstrass M-Test (Proposition 7.9) to deduce that the power series $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ converges uniformly on the disk $\{z \in \mathbb{C} : |z-z_0| < R\}$ of radius R about z_0. It then follows from Theorem 6.12 that the restriction of the function s to this disk is continuous on the disk, and, in particular, is continuous around z_1. We deduce that the function s is continuous throughout the complex plane when $R_0 = +\infty$, and is continuous inside the circle of convergence when $R_0 < +\infty$, as required. ∎

A power series with finite radius of convergence will converge everywhere within its circle of convergence, and will diverge everywhere outside this circle. However Theorem 7.10 provides no information concerning the behaviour of the power series on the circle of convergence itself.

We can define exponential and trigonometric functions of a complex variable by means of power series. Let

$$\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}, \qquad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \qquad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Now a straightforward application of the Ratio Test shows that these power series have infinite radius of convergence. It follows from Theorem 7.10 that the functions exp, sin and cos defined in this fashion are continuous functions defined over the whole of $\mathbb{C}$. Moreover they agree with the usual exponential, sine and cosine functions for all real values of z; this follows on applying Taylor's Theorem to these functions. Note that $\exp(iz) = \cos(z) + i\sin(z)$ for all $z \in \mathbb{C}$, and thus

$$\sin(z) = \frac{1}{2i}(\exp(iz) - \exp(-iz)), \qquad \cos(z) = \frac{1}{2}(\exp(iz) + \exp(-iz)).$$

8 Euclidean Spaces, Continuity, and Open Sets

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Lemma 8.1 (Schwarz' Inequality) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$. *Then* $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.

Proof We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}\cdot\mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}\cdot\mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x}\cdot\mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x}\cdot\mathbf{y})^2 + (\mathbf{x}\cdot\mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

It follows easily from Schwarz' Inequality that $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. For

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

It follows that

$$|\mathbf{z} - \mathbf{x}| \leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $|\mathbf{z}|$ of $\mathbb{R}^n$. This important inequality is known as the *Triangle Inequality*. It expresses the geometric fact the the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some natural number N such that $|\mathbf{p} - \mathbf{x}_j| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j\to+\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 8.2 *Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.*

Proof Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$, where $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_j$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist natural numbers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be a *Cauchy sequence* if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some natural number N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ whenever $j \geq N$ and $k \geq N$.

Lemma 8.3 *A sequence of points in $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of $\mathbb{R}^n$ converging to some point $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. If $j \geq N$ and $k \geq N$ then

$$|\mathbf{x}_j - \mathbf{x}_k| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{p} - \mathbf{x}_k| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

by the Triangle Inequality. Thus every convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

Now let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then the ith components of the elements of this sequence constitute a Cauchy sequence of real numbers. This Cauchy sequence must converge to some real number p_i, by Cauchy's Criterion for Convergence (Theorem 6.6). It follows from Lemma 8.2 that the Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. ∎

Definition Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 8.4 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f: X \to Y$ and $g: Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f: X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 8.5 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some natural number N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 8.6 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ is continuous if and only if its components are continuous.*

Proof Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 8.4. Thus if f is continuous, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 8.7 *The functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $p(x, y) = xy$ are continuous.*

Proof Let $(u, v) \in \mathbb{R}^2$. We first show that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v).

Next we show that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Now

$$p(x, y) - p(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|p(x, y) - p(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ is given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|p(x, y) - p(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Proposition 8.8 *Let X be a subset of $\mathbb{R}^n$, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = p \circ h$, where $h: X \to \mathbb{R}^2$, $s: \mathbb{R}^2 \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $p(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 8.6, Lemma 8.7 and Lemma 8.4 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Consider the function $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 8.8. It then follows from Proposition 8.6 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

8.1 Open Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X given by

$$B_X(\mathbf{p}, r) \equiv \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in the case when V is the empty set.)

In particular, a subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$.

Example Let $H = \{(x,y,z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u,v,w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x,y,z) to the point (u,v,w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x,y,z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}, \qquad \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$ of radius δ about $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ about the point $(0, 0, 1)$ is not a subset of N.

Lemma 8.9 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 8.10 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 8.11 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are both open in X;*

(ii) *the union of any collection of open sets in X is itself open in X;*

(iii) *the intersection of any* finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x,y,z) \in \mathbb{R}^3 : (x-n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n,0,0)$ for all integers n.

Example For each natural number k, let

$$V_k = \{(x,y,z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all natural numbers k is the set $\{(0,0,0)\}$, and thus the intersection of the sets V_k for all natural numbers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Lemma 8.12 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some natural number N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some natural number N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 8.9. Therefore there exists some natural number N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some natural number N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

8.2 Closed Sets

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x,y,z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x,y,z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x,y,z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 8.9 and Lemma 8.10 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets). The following result therefore follows directly from Proposition 8.11.

Proposition 8.13 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are both closed in X;*

(ii) *the intersection of any collection of closed sets in X is itself closed in X;*

(iii) *the union of any* finite *collection of closed sets in X is itself closed in X.*

Lemma 8.14 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 8.12 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

8.3 Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f: X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{u}) - f(\mathbf{p})| < \varepsilon$

for all points $\mathbf{u}$ of X satisfying $|\mathbf{u} - \mathbf{p}| < \delta$. Thus the function $f: X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

Given any function $f: X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 8.15 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f: X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof Suppose that $f: X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f: X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_X(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 8.9, hence $f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f: X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.