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1 The Real Number System

1.1 Sets

A *set* is a collection of objects. These objects are referred to as the *elements* of the set. One can specify a set by enclosing a list of suitable objects within braces. Thus, for example, $\{1, 2, 3, 7\}$ denotes the set whose elements are the numbers 1, 2, 3 and 7. If x is an element of some set X then we denote this fact by writing $x \in X$. Conversely, if x is not an element of the set X then we write $x \notin X$. We denote by $\emptyset$ the *empty set*, which is defined to be the set with no elements.

We denote by $\mathbb{N}$ the set $\{1, 2, 3, 4, 5 \ldots\}$ of all *natural numbers*, and we denote by $\mathbb{Z}$ the set

$$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

of all *integers* (or 'whole numbers'). We denote by $\mathbb{Q}$ the set of *rational numbers* (i.e., numbers of the form p/q where p and q are integers and $q \neq 0$), and we denote be $\mathbb{R}$ and $\mathbb{C}$ the sets of real numbers and complex numbers respectively.

If X and Y are sets then the *union* $X \cup Y$ of X and Y is defined to be the set of all elements that belong either to X or to Y (or to both), the *intersection* $X \cap Y$ of X and Y is defined to be the set of all elements that belong to both X and Y, and the *difference* $X \setminus Y$ of X and Y is defined to be the set of all elements that belong to X but do not belong to Y. Thus, for example, if

$$X = \{2, 4, 6, 8\}, \qquad Y = \{3, 4, 5, 6, 7\}$$

then

$$X \cup Y = \{2, 3, 4, 5, 6, 7, 8\}, \qquad X \cap Y = \{4, 6\},$$

$$X \setminus Y = \{2, 8\}, \qquad Y \setminus X = \{3, 5, 7\}.$$

If X and Y are sets, and if every element of X is also an element of Y then we say that X is a *subset* of Y, and we write $X \subset Y$. We use the notation $\{y \in Y : P(y)\}$ to denote the subset of a given set Y consisting of all elements y of Y with some given property $P(y)$. Thus for example $\{n \in \mathbb{Z} : n > 0\}$ denotes the set of all integers n satisfying $n > 0$ (i.e., the set $\mathbb{N}$ of all natural numbers).

1.2 Rational and Irrational Numbers

Rational numbers are numbers that can be expressed as fractions of the form p/q, where p and q are integers (i.e., 'whole numbers') and $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$. Operations of addition, subtraction, multiplication and division are defined on $\mathbb{Q}$ in the usual manner. In addition the set of rational numbers is ordered.

There are however certain familiar numbers which cannot be represented in the form p/q, where p and q are integers. These include $\sqrt{2}$, $\sqrt{3}$, π and e. Such numbers are referred to as *irrational numbers.* The irrationality of $\sqrt{2}$ is an immediate consequence of the following famous result, which was discovered by the Ancient Greeks.

Proposition 1.1 *There do not exist non-zero integers p and q with the property that $p^2 = 2q^2$.*

Proof Let us suppose that there exist non-zero integers p and q with the property that $p^2 = 2q^2$. We show that this leads to a contradiction. Without loss of generality we may assume that p and q are not both even (since if both p and q were even then we could replace p and q by $p/2^k$ and $q/2^k$ respectively, where k is the largest natural number with the property that 2^k divides both p and q). Now $p^2 = 2q^2$, hence p^2 is even. It follows from this that p is even (since the square of an odd integer is odd). Therefore $p = 2r$ for some integer r. But then $2q^2 = 4r^2$, so that $q^2 = 2r^2$. Therefore q^2 is even, and hence q is even. We have thus shown that both p and q are even. But this contradicts our assumption that p and q are not both even. This contradiction shows that there cannot exist integers p and q with the property that $p^2 = 2q^2$, and thus proves that $\sqrt{2}$ is an irrational number. ∎

This result shows that the rational numbers are not sufficient for the purpose of representing lengths arising in familiar Euclidean geometry. Indeed consider the right-angled isosceles triangle whose short sides are q units long. Then the hypotenuse is $\sqrt{2}q$ units long, by Pythagoras' Theorem. Proposition 1.1 shows that it is not possible to find a unit of length for which the two short sides of this right-angled isosceles triangle are q units long and the hypotenuse is p units long, where both p and q are integers. We must therefore enlarge the system of rational numbers to obtain a number system which contains irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, π and e, and which is capable of representing the lengths of line segments and similar quantities arising in geometry and physics. The rational and irrational numbers belonging to this number system are known as *real numbers*.

1.3 The Real Number System

The system of real numbers, denoted by $\mathbb{R}$, is an ordered set on which are defined appropriate operations of addition and multiplication. The system of real numbers is fully characterized by an axiom system consisting of the 15 axioms (listed below) which describe the algebraic structure and ordering of the real numbers, together with one further axiom, known as the *Least Upper Bound Axiom*, which distinguishes the real number system from other number systems such as the rational number system. The 15 axioms describing the algebraic and ordering properties of the real number system are as follows:

1. if x and y are real numbers then their sum $x + y$ is also a real number,
2. (*the Commutative Law for addition*) $x + y = y + x$ for all real numbers x and y,
3. (*the Associative Law for addition*) $(x + y) + z = x + (y + z)$ for all real numbers x, y and z,
4. there exists a (necessarily unique) real number, denoted by 0, with the property that $x + 0 = x = 0 + x$ for all real numbers x,
5. for each real number x there exists some (necessarily unique) real number $-x$ with the property that $x + (-x) = 0 = (-x) + x$,
6. if x and y are real numbers then their product xy is also a real number,
7. (*the Commutative Law for multiplication*) $xy = yx$ for all real numbers x and y,
8. (*the Associative Law for multiplication*) $(xy)z = x(yz)$ for all real numbers x, y and z,
9. there exists a (necessarily unique) real number, denoted by 1, with the property that $x1 = x = 1x$ for all real numbers x, and moreover $1 \neq 0$,

10. for each real number x satisfying $x \neq 0$ there exists some (necessarily unique) real number x^{-1} with the property that $xx^{-1} = 1 = x^{-1}x$,
11. (*the Distributive Law*) $x(y + z) = (xy) + (xz)$ for all real numbers x y and z,
12. (*the Trichotomy Law*) if x and y are real numbers then one and only one of the three statements $x < y$, $x = y$ and $y < x$ is true,
13. if x, y and z are real numbers and if $x < y$ and $y < z$ then $x < z$,
14. if x, y and z are real numbers and if $x < y$ then $x + z < y + z$,
15. if x and y are real numbers which satisfy $0 < x$ and $0 < y$ then $0 < xy$,

The operations of subtraction and division are defined in terms of addition and multiplication in the obvious fashion: $x - y = x + (-y)$ for all real numbers x and y, and $x/y = xy^{-1}$ provided that $y \neq 0$. The *absolute value* $|x|$ of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all real numbers x and that $|x| = 0$ if and only if $x = 0$. Also $|x + y| \leq |x| + |y|$ and $|xy| = |x||y|$ for all real numbers x and y.

Let D be a subset of $\mathbb{R}$. A real number u is said to be an *upper bound* of the set D if $x \leq u$ for all $x \in D$. The set D is said to be *bounded above* if such an upper bound exists.

Definition Let D be some set of real numbers which is bounded above. A real number s is said to be the *least upper bound* (or *supremum*) of D (denoted by $\sup D$) if s is an upper bound of D and $s \leq u$ for all upper bounds u of D.

Example The real number 2 is the least upper bound of the sets $\{x \in \mathbb{R} : x \leq 2\}$ and $\{x \in \mathbb{R} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The axioms (1)–(15) listed above describing the algebraic and ordering properties of the real number system are not in themselves sufficient to fully characterize the real number system. (Indeed any property of real numbers that could be derived solely from these axioms would be equally valid for rational numbers.) We require in addition the following axiom:—

> *the Least Upper Bound Axiom*: if D is any non-empty subset of $\mathbb{R}$ which is bounded above then there exists a *least upper bound* $\sup D$ for the set D.

A *lower bound* of a set D of real numbers is a real number l with the property that $l \leq x$ for all $x \in D$. A set D of real numbers is said to be *bounded below* if such a lower bound exists. If D is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf D$ of the set D. Indeed $\inf D = -\sup\{x \in \mathbb{R} : -x \in D\}$.

Remark We have simply listed above a complete set of axioms for the real number system. We have not however proved the existence of a system of real numbers satisfying these axioms. There are in fact several constructions of the real number system: one of the most popular of these is the representation of real numbers as *Dedekind sections* of the set of rational numbers. For an account of the this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak.

1.4 Intervals

Given real numbers a and b satisfying $a \leq b$, we define

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

If $a < b$ then we define

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}, \qquad [a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

For each real number c, we also define

$$[c, +\infty) = \{x \in \mathbb{R} : c \leq x\}, \qquad (c, +\infty) = \{x \in \mathbb{R} : c < x\},$$

$$(-\infty, c] = \{x \in \mathbb{R} : x \leq c\}, \qquad (-\infty, c) = \{x \in \mathbb{R} : x < c\}.$$

All these subsets of $\mathbb{R}$ are referred to as *intervals*. An *interval* I may be defined as a non-empty set of real numbers with the following property: if s, t and u are real numbers satisfying $s < t < u$ and if s and u both belong to the interval I then t also belongs to the interval I. Using the Least Upper Bound Axiom, one can prove that every interval in $\mathbb{R}$ is either one of the intervals defined above, or else is the whole of $\mathbb{R}$.

2 Infinite Sequences of Real Numbers

2.1 Convergence

An *infinite sequence* of real numbers is a sequence of the form $a_1, a_2, a_3, \ldots$, where each a_n is a real number. (More formally, one can view an infinite sequence of real numbers as a function from $\mathbb{N}$ to $\mathbb{R}$ which sends each natural number n to some real number a_n.)

Definition A sequence $a_1, a_2, a_3, \ldots$ of real numbers is said to *converge* to some real number l if and only if the following criterion is satisfied:

> given any real number ε satisfying $\varepsilon > 0$, there exists some natural number N such that $|a_n - l| < \varepsilon$ for all n satisfying $n \geq N$.

If the sequence $a_1, a_2, a_3, \ldots$ converges to the *limit* l then we denote this fact by writing '$a_n \to l$ as $n \to +\infty$', or by writing '$\lim\limits_{n\to+\infty} a_n = l$'.

Example A straightforward application of the definition of convergence shows that $1/n \to 0$ as $n \to +\infty$. Indeed suppose that we are given any real number ε satisfying $\varepsilon > 0$. If we pick some natural number N large enough to satisfy $N > 1/\varepsilon$ then $|1/n| < \varepsilon$ for all natural numbers n satisfying $n \geq N$, as required.

Example We show that $(-1)^n/n^2 \to 0$ as $n \to +\infty$. Indeed, given any real number ε number satisfying $\varepsilon > 0$, we can find some natural number N satisfying $N^2 > 1/\varepsilon$. If $n \geq N$ then $|(-1)^n/n^2| < \varepsilon$, as required.

Example The infinite sequence $a_1, a_2, a_3, \ldots$ defined by $a_n = n$ is not convergent. To prove this formally, we suppose that it were the case that $\lim\limits_{n\to+\infty} a_n = l$ for some real number l, and derive from this a contradiction. On setting $\varepsilon = 1$ (say) in the formal definition of convergence, we would deduce that there would exist some natural number N such that $|a_n - l| < 1$ for all $n \geq N$. But then $a_n < l + 1$ for all $n \geq N$, which is impossible. Thus the sequence cannot converge.

Example The infinite sequence $u_1, u_2, u_3, \ldots$ defined by $u_n = (-1)^n$ is not convergent. To prove this formally, we suppose that it were the case that $\lim\limits_{n\to+\infty} u_n = l$ for some real number l. On setting $\varepsilon = \frac{1}{2}$ in the criterion for convergence, we would deduce the existence of some natural number N such that $|u_n - l| < \frac{1}{2}$ for all $n \geq N$. But then

$$|u_n - u_{n+1}| \leq |u_n - l| + |l - u_{n+1}| < \tfrac{1}{2} + \tfrac{1}{2} = 1$$

for all $n \geq N$, contradicting the fact that $u_n - u_{n+1} = \pm 2$ for all n. Thus the sequence cannot converge.

Definition We say that an infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $a_n \leq B$ for all n. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $a_n \geq A$ for all n. A sequence is said to be *bounded* if it is bounded above and bounded below, so that there exist real numbers A and B such that $A \leq a_n \leq B$ for all n.

Lemma 2.1 *Every convergent sequence of real numbers is bounded.*

Proof Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers converging to some real number l. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some natural number N such that $|a_n - l| < 1$ for all $n \geq N$. But then $A \leq a_n \leq B$ for all n, where A is the minimum of $a_1, a_2, \ldots, a_{N-1}$ and $l - 1$, and B is the maximum of $a_1, a_2, \ldots, a_{N-1}$ and $l + 1$. ∎

Proposition 2.2 *Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3,$ be convergent infinite sequences of real numbers. Then the sum, difference and product of these sequences are convergent, and*

$$\begin{aligned}
\lim_{n\to+\infty} (a_n + b_n) &= \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n, \\
\lim_{n\to+\infty} (a_n - b_n) &= \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n, \\
\lim_{n\to+\infty} (a_n b_n) &= \left(\lim_{n\to+\infty} a_n\right)\left(\lim_{n\to+\infty} b_n\right).
\end{aligned}$$

If in addition $b_n \neq 0$ for all n and $\lim_{n\to+\infty} b_n \neq 0$, then the quotient of the sequences (a_n) and (b_n) is convergent, and

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to+\infty} a_n}{\lim\limits_{n\to+\infty} b_n}.$$

Proof Throughout this proof let $l = \lim_{n\to+\infty} a_n$ and $m = \lim_{n\to+\infty} b_n$.

First we prove that $a_n + b_n \to l + m$ as $n \to +\infty$. Let ε be any given real number satisfying $\varepsilon > 0$. We must show that there exists some natural number N such that $|a_n + b_n - (l + m)| < \varepsilon$ whenever $n \geq N$. Now $a_n \to l$ as $n \to +\infty$, and therefore, given any $\varepsilon_1 > 0$, there exists some natural number N_1 with the property that $|a_n - l| < \varepsilon_1$ whenever $n \geq N_1$. In particular, there exists a natural number N_1 with the property that $|a_n - l| < \frac{1}{2}\varepsilon$ whenever $n \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some natural number N_2 such that $|b_n - m| < \frac{1}{2}\varepsilon$ whenever $n \geq N_2$. Let N be the maximum of N_1 and N_2. If $n \geq N$ then

$$\begin{aligned} |a_n + b_n - (l+m)| &= |(a_n - l) + (b_n - m)| \leq |a_n - l| + |b_n - m| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $a_n + b_n \to l + m$ as $n \to +\infty$.

Let c be some real number. We show that $cb_n \to cm$ as $n \to +\infty$. The case when $c = 0$ is trivial. Suppose that $c \neq 0$. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $|b_n - m| < \varepsilon/|c|$ whenever $n \geq N$. But then $|cb_n - cm| = |c||b_n - m| < \varepsilon$ whenever $n \geq N$. Thus $cb_n \to cm$ as $n \to +\infty$.

If we combine this result, for $c = -1$, with the previous result, we see that $-b_n \to -m$ as $n \to +\infty$, and therefore $a_n - b_n \to l - m$ as $n \to +\infty$.

Next we show that if $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are infinite sequences, and if $u_n \to 0$ and $v_n \to 0$ as $n \to +\infty$, then $u_n v_n \to 0$ as $n \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist natural numbers N_1 and N_2 such that $|u_n| < \sqrt{\varepsilon}$ whenever $n \geq N_1$ and $|v_n| < \sqrt{\varepsilon}$ whenever $n \geq N_2$. Let N be the maximum of N_1 and N_2. If $n \geq N$ then $|u_n v_n| < \varepsilon$. We deduce that $u_n v_n \to 0$ as $n \to +\infty$.

We can apply this result with $u_n = a_n - l$ and $v_n = b_n - m$ for all natural numbers n. Using the results we have already obtained, we see that

$$\begin{aligned} 0 &= \lim_{n\to+\infty} (u_n v_n) = \lim_{n\to+\infty} (a_n b_n - a_n m - l b_n + lm) \\ &= \lim_{n\to+\infty} (a_n b_n) - m \lim_{n\to+\infty} a_n - l \lim_{n\to+\infty} b_n + lm = \lim_{n\to+\infty} (a_n b_n) - lm. \end{aligned}$$

Thus $a_n b_n \to lm$ as $n \to +\infty$.

Next we show that if $w_1, w_2, w_3, \ldots$ is an infinite sequence of non-zero real numbers, and if $w_n \to 1$ as $n \to +\infty$ then $1/w_n \to 1$ as $n \to +\infty$. Let $\varepsilon > 0$ be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some natural number N such that $|w_n - 1| < \varepsilon_0$ whenever $n \geq N$. Thus if $n \geq N$ then $|w_n - 1| < \frac{1}{2}\varepsilon$ and $\frac{1}{2} < w_n < \frac{3}{2}$. But then

$$\left|\frac{1}{w_n} - 1\right| = \left|\frac{1 - w_n}{w_n}\right| = \frac{|w_n - 1|}{|w_n|} < 2|w_n - 1| < \varepsilon.$$

We deduce that $1/w_n \to 1$ as $n \to +\infty$.

Finally suppose that $\lim_{n\to+\infty} a_n = l$ and $\lim_{n\to+\infty} b_n = m$, where $m \neq 0$. Let $w_n = b_n/m$. Then $w_n \to 1$ as $n \to +\infty$, and hence $1/w_n \to 1$ as $n \to +\infty$.

We see therefore that $m/b_n \to 1$, and thus $1/b_n \to 1/m$, as $n \to +\infty$. The result we have already obtained for products of sequences then enables us to deduce that $a_n/b_n \to l/m$ as $n \to +\infty$. ∎

Example We shall show that if $s_n \to 2$ as $n \to +\infty$, where $s_n = \dfrac{6n^2 - 4n}{3n^2 + 7}$ for all natural numbers n. Now neither $6n^2 - 4n$ nor $3n^2 + 7$ converges to any (finite) limit as $n \to +\infty$; and therefore we cannot directly apply the result in Proposition 2.2 concerning the convergence of the quotient of two convergent sequences. However on dividing both the numerator and the denominator of the fraction defining s_n by n^2, we see that

$$s_n = \frac{6n^2 - 4n}{3n^2 + 7} = \frac{6 - \dfrac{4}{n}}{3 + \dfrac{7}{n^2}}.$$

Moreover $6 - \dfrac{4}{n} \to 6$ and $3 + \dfrac{7}{n^2} \to 3$ as $n \to +\infty$, and therefore, on applying Proposition 2.2, we see that

$$\lim_{n\to+\infty} \frac{6n^2 - 4n}{3n^2 + 7} = \lim_{n\to+\infty} \frac{6 - \dfrac{4}{n}}{3 + \dfrac{7}{n^2}} = \frac{\lim\limits_{n\to+\infty} \left(6 - \dfrac{4}{n}\right)}{\lim\limits_{n\to+\infty} \left(3 + \dfrac{7}{n^2}\right)} = \frac{6}{3} = 2.$$

2.2 Monotonic Sequences

An infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers is said to be *strictly increasing* if $a_{n+1} > a_n$ for all n, *strictly decreasing* if $a_{n+1} < a_n$ for all n, *non-decreasing* if $a_{n+1} \geq a_n$ for all n, or *non-increasing* if $a_{n+1} \leq a_n$ for all n. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 2.3 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $a_1, a_2, a_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound l for the set $\{a_n : n \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let $\varepsilon > 0$ be given. We must show that there exists some natural number N such that $|a_n - l| < \varepsilon$ whenever $n \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{a_n : n \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some natural number N such that $a_N > l - \varepsilon$. But then $l - \varepsilon < a_n \leq l$ whenever $n \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|a_n - l| < \varepsilon$ whenever $n \geq N$. Therefore $a_n \to l$ as $n \to +\infty$, as required.

If the sequence $a_1, a_2, a_3, \ldots$ is non-increasing and bounded below then the sequence $-a_1, -a_2, -a_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $a_1, a_2, a_3, \ldots$ is also convergent. ∎

Example Let $a_1 = 2$ and

$$a_{n+1} = a_n - \frac{a_n^2 - 2}{2a_n}$$

for all natural numbers n. Now

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n} \text{ and } a_{n+1}^2 = a_n^2 - (a_n^2 - 2) + \left(\frac{a_n^2 - 2}{2a_n}\right)^2 = 2 + \left(\frac{a_n^2 - 2}{2a_n}\right)^2.$$

It therefore follows by induction on n that $a_n > 0$ and $a_n^2 > 2$ for all natural numbers n. But then $a_{n+1} < a_n$ for all n, and thus the sequence $a_1, a_2, a_3, \ldots$ is decreasing and bounded below. It follows from Theorem 2.3 that this sequence converges to some real number α. Also $a_n > 1$ for all n (since $a_n > 0$ and $a_n^2 > 2$), and therefore $\alpha \geq 1$. But then, on applying Proposition 2.2, we see that

$$\alpha = \lim_{n \to +\infty} a_{n+1} = \lim_{n \to +\infty} \left(a_n - \frac{a_n^2 - 2}{2a_n}\right) = \alpha - \frac{\alpha^2 - 2}{2\alpha}.$$

Thus $\alpha^2 = 2$, and so $\alpha = \sqrt{2}$.

2.3 Subsequences and the Bolzano-Weierstrass Theorem

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this sequence is a sequence of the form $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$, where $n_1, n_2, n_3, \ldots$ are natural numbers satisfying $n_1 < n_2 < n_3 < \cdots$. Thus, for example, $a_2, a_4, a_6, \ldots$ and $a_1, a_4, a_9, \ldots$ are subsequences of the given sequence.

Theorem 2.4 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers, and let

$$S = \{n \in \mathbb{N} : a_n \geq a_k \text{ for all } k \geq n\}$$

(i.e., S is the set of all natural numbers n with the property that a_n is greater than or equal to all the succeeding members of the sequence).

First let us suppose that the set S is infinite. Arrange the elements of S in increasing order so that $S = \{n_1, n_2, n_3, n_4, \ldots\}$, where $n_1 < n_2 < n_3 < n_4 < \cdots$. It follows from the manner in which the set S was defined that $a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq a_{n_4} \geq \cdots$. Thus $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 2.3 that $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S is finite. Choose a natural number n_1 which is greater than every natural number belonging to S. Then n_1 does not belong to S. Therefore there must exist some natural number n_2 satisfying $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. Moreover n_2 does not belong to S (since n_2 is greater than n_1 and n_1 is greater than every natural number belonging to S). Therefore there must exist some natural number n_3 satisfying $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 2.3. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

3 Limits and Continuity

3.1 Limits of Functions of a Real Variable

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number (which may or may not belong to D). We say that s is a *limit point* of D if, given any $\delta > 0$, there exists some $x \in D$ which satisfies $0 < |x - s| < \delta$.

We now define the *limit* of a real-valued function at any limit point of the domain of that function.

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s be a limit point of D. A real number l is said to be the *limit* of the function f as x tends to s in D if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for all $x \in D$ satisfying $0 < |x - s| < \delta$.

If l is the limit of $f(x)$ as x tends to s, for some s, then we denote this fact either by writing '$f(x) \to l$ as $x \to s$' or by writing '$\lim_{x \to s} f(x) = l$'.

Note that $\lim_{x \to s} f(x) = l$ if and only if $\lim_{h \to 0} f(s+h) = l$: this follows directly from the definition given above.

Lemma 3.1 *Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s be a limit point of D. Then the limit $\lim_{x \to s} f(x)$, if it exists, is unique.*

Proof Suppose that $\lim_{x \to s} f(x) = l$ and $\lim_{x \to s} f(x) = m$. We must show that $l = m$. Let $\varepsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - l| < \varepsilon$ whenever $x \in D$ satisfies $0 < |x - s| < \delta_1$ and $|f(x) - m| < \varepsilon$ whenever $x \in D$ satisfies $0 < |x - s| < \delta_2$. Choose $x \in D$ satisfying $0 < |x - s| < \delta$, where δ is the minimum of δ_1 and δ_2. (This is possible since s is a limit point of D.) Then $|f(x) - l| < \varepsilon$ and $|f(x) - m| < \varepsilon$, and hence

$$|l - m| \leq |l - f(x)| + |f(x) - m| < 2\varepsilon$$

by the Triangle Inequality. Since $|l - m| < 2\varepsilon$ for all $\varepsilon > 0$, we conclude that $l = m$, as required. ∎

Example We show that $\lim_{x \to 0} \frac{1}{4}x^2 = 0$. Let $\varepsilon > 0$ be given. Suppose that we choose $\delta = 2\sqrt{\varepsilon}$, for example. If $0 < |x| < \delta$ then $|\frac{1}{4}x^2| < \frac{1}{4}\delta^2 = \varepsilon$, as required.

Example We show that $\lim_{x \to 0} 3x \cos(1/x) = 0$. Let $\varepsilon > 0$ be given. Choose $\delta = \frac{1}{3}\varepsilon$. Then $\delta > 0$. Moreover if $0 < |x| < \delta$ then $|3x \cos(1/x)| < \varepsilon$, as required.

Proposition 3.2 *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be functions defined over some subset D of $\mathbb{R}$. Let s be a limit point of D. Suppose that $\lim_{x \to s} f(x)$ and $\lim_{x \to s} g(x)$ exist. Then $\lim_{x \to s} (f(x) + g(x))$, $\lim_{x \to s} (f(x) - g(x))$ and $\lim_{x \to s} (f(x)g(x))$ exist, and*

$$\begin{aligned}
\lim_{x \to s} (f(x) + g(x)) &= \lim_{x \to s} f(x) + \lim_{x \to s} g(x), \\
\lim_{x \to s} (f(x) - g(x)) &= \lim_{x \to s} f(x) - \lim_{x \to s} g(x), \\
\lim_{x \to s} (f(x)g(x)) &= \lim_{x \to s} f(x) \lim_{x \to s} g(x).
\end{aligned}$$

If in addition $g(x) \neq 0$ *for all* $x \in D$ *and* $\lim_{x\to s} g(x) \neq 0$, *then* $\lim_{x\to s} f(x)/g(x)$ *exists, and*

$$\lim_{x\to s} \frac{f(x)}{g(x)} = \frac{\lim_{x\to s} f(x)}{\lim_{x\to s} g(x)}.$$

Proof Let $l = \lim_{x\to s} f(x)$ and $m = \lim_{x\to s} g(x)$.

First we prove that $\lim_{x\to s}(f(x) + g(x)) = l + m$. Let $\varepsilon > 0$ be given. We must prove that there exists some $\delta > 0$ such that $|f(x)+g(x)-(l+m)| < \varepsilon$ for all $x \in D$ satisfying $0 < |x - s| < \delta$. Now there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - l| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $0 < |x - s| < \delta_1$, and $|g(x) - m| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $0 < |x - s| < \delta_2$, since $l = \lim_{x\to s} f(x)$ and $m = \lim_{x\to s} g(x)$. Let δ be the minimum of δ_1 and δ_2. If $x \in D$ satisfies $0 < |x - s| < \delta$ then $|f(x) - l| < \frac{1}{2}\varepsilon$ and $|g(x) - m| < \frac{1}{2}\varepsilon$, and hence

$$|f(x) + g(x) - (l + m)| \leq |f(x) - l| + |g(x) - m| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

This shows that $\lim_{x\to s} (f(x) + g(x)) = l + m$.

Let c be some real number. We show that $\lim_{x\to s} (cg(x)) = cm$. The case when $c = 0$ is trivial. Suppose that $c \neq 0$. Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|g(x) - m| < \varepsilon/|c|$ whenever $0 < |x - s| < \delta$. But then $|cg(x) - cm| = |c||g(x) - m| < \varepsilon$ whenever $0 < |x - s| < \delta$. Thus $\lim_{x\to s} (cg(x)) = cm$.

If we combine this result, for $c = -1$, with the previous result, we see that $\lim_{x\to s} (-g(x)) = -m$, and therefore $\lim_{x\to s} (f(x) - g(x)) = l - m$.

Next we show that if $p\colon D \to \mathbb{R}$ and $q\colon D \to \mathbb{R}$ are functions with the property that $\lim_{x\to s} p(x) = \lim_{x\to s} q(x) = 0$, then $\lim_{x\to s} (p(x)q(x)) = 0$. Let $\varepsilon > 0$ be given. Then there exist real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|p(x)| < \sqrt{\varepsilon}$ whenever $0 < |x - s| < \delta_1$ and $|q(x)| < \sqrt{\varepsilon}$ whenever $0 < |x - s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $0 < |x - s| < \delta$ then $|p(x)q(x)| < \varepsilon$. We deduce that $\lim_{x\to s} (p(x)q(x)) = 0$.

We can apply this result with $p(x) = f(x) - l$ and $q(x) = g(x) - m$ for all $x \in D$. Using the results we have already obtained, we see that

$$\begin{aligned} 0 &= \lim_{x\to s}(p(x)q(x)) = \lim_{x\to s}(f(x)g(x) - f(x)m - lg(x) + lm) \\ &= \lim_{x\to s}(f(x)g(x)) - m \lim_{x\to s} f(x) - l \lim_{x\to s} g(x) + lm = \lim_{x\to s}(f(x)g(x)) - lm. \end{aligned}$$

Thus $\lim_{x\to s} (f(x)g(x)) = lm$.

Next we show that if $h: D \to R$ is a function that is non-zero throughout D, and if $\lim_{x \to s} h(x) \to 1$ then $\lim_{x \to s} (1/h(x)) = 1$. Let $\varepsilon > 0$ be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some $\delta > 0$ such that $|h(x) - 1| < \varepsilon_0$ whenever $0 < |x - s| < \delta$. Thus if $0 < |x - s| < \delta$ then $|h(x) - 1| < \frac{1}{2}\varepsilon$ and $\frac{1}{2} < h(x) < \frac{3}{2}$. But then

$$\left| \frac{1}{h(x)} - 1 \right| = \left| \frac{h(x) - 1}{h(x)} \right| = \frac{|h(x) - 1|}{|h(x)|} < 2|h(x) - 1| < \varepsilon.$$

We deduce that $\lim_{x \to s} 1/h(x) = 1$. If we apply this result with $h(x) = g(x)/m$, where $m \neq 0$, we deduce that $\lim_{x \to s} m/g(x) = 1$, and thus $\lim_{x \to s} 1/g(x) = 1/m$. The result we have already obtained for products of functions then enables us to deduce that $\lim_{x \to s} (f(x)/g(x)) \to l/m$.

3.2 Continuous Functions of a Real Variable

Definition Let D be a subset of $\mathbb{R}$, and let $f: D \to \mathbb{R}$ be a real-valued function on D. Let s be a point of D. The function f is said to be *continuous* at s if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. If f is continuous at every point of D then we say that f is continuous on D.

Example Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function f is not continuous at 0. To prove this formally we note that when $0 < \varepsilon \leq 1$ there does not exist any $\delta > 0$ with the property that $|f(x) - f(0)| < \varepsilon$ for all x satisfying $|x| < \delta$ (since $|f(x) - f(0)| = 1$ for all $x > 0$).

Example Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We show that this function is not continuous at 0. Suppose that ε is chosen to satisfy $0 < \varepsilon < 1$. No matter how small we choose δ, where $\delta > 0$, we can always find $x \in \mathbb{R}$ for which $|x| < \delta$ and $|f(x) - f(0)| \geq \varepsilon$. Indeed, given any $\delta > 0$, we can choose some integer n large enough to ensure that $0 < x_n < \delta$, where x_n satisfies $1/x_n = (4n+1)\pi/2$. Moreover $f(x_n) = 1$. This shows that the criterion defining the concept of continuity is not satisfied at $x = 0$.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 3x \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that the function f is continuous at 0. To prove this, we must apply the definition of continuity directly. Suppose we are given any real number ε satisfying $\varepsilon > 0$. If $\delta = \frac{1}{3}\varepsilon$ then $|f(x)| \leq 3|x| < \varepsilon$ for all real numbers x satisfying $|x| < \delta$, as required.

The following lemma describes the relationship between limits and continuity.

Lemma 3.3 *Let D be a subset of $\mathbb{R}$, and let $s \in D$.*

(i) *Suppose that s is a limit point of D. Then a function $f: D \to \mathbb{R}$ with domain D is continuous at s if and only if $\lim_{x \to s} f(x) = f(s)$;*

(ii) *Suppose that s is not a limit point of D. Then every function $f: D \to \mathbb{R}$ with domain D is continuous at s.*

Proof If s is a limit point of D belonging to D then the required result follows immediately on comparing the formal definition of the limit of a function with the formal definition of continuity (since the condition $|f(x) - f(s)| < \varepsilon$ is automatically satisfied for any $\varepsilon > 0$ when $x = s$).

Suppose that s is not a limit point of D. Then there exists some $\delta > 0$ such that the only element x of D satisfying $|x - s| < \delta$ is s itself. The definition of continuity is therefore satisfied trivially at s by any function $f: D \to \mathbb{R}$ with domain D. ∎

Given functions $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ defined over some subset D of $\mathbb{R}$, we denote by $f + g$, $f - g$, $f \cdot g$ and $|f|$ the functions on D defined by

$$(f + g)(x) = f(x) + g(x), \qquad (f - g)(x) = f(x) - g(x),$$

$$(f \cdot g)(x) = f(x)g(x), \qquad |f|(x) = |f(x)|.$$

Proposition 3.4 *Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be functions defined over some subset D of $\mathbb{R}$. Suppose that f and g are continuous at some point s of D. Then the functions $f + g$, $f - g$ and $f \cdot g$ are also continuous at s. If moreover the function g is everywhere non-zero on D then the function f/g is continuous at s.*

Proof This result follows directly from Proposition 3.2, using the fact that a function $f\colon D \to \mathbb{R}$ is continuous at a limit point s of D belonging to D if and only if $\lim_{x \to s} f(x) = f(s)$ (Lemma 3.3). ∎

Remark Proposition 3.4 can also be proved directly from the formal definition of continuity by a straightforward adaptation of the proof of Proposition 3.2. Indeed suppose that $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are continuous at s, where $s \in D$. We show that $f + g$ is continuous at s. Let $\varepsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - f(s)| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta_1$, and $|g(x) - g(s)| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $|x - s| < \delta$ then

$$|f(x) + g(x) - (f(s) + g(s))| \leq |f(x) - f(s)| + |g(x) - g(s)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

showing that $f + g$ is continuous at s. The proof of Proposition 3.2 can be adapted in a similar fashion to show that $f - g$, $f \cdot g$ and f/g are also continuous at s.

Proposition 3.5 *Let $f\colon D \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$ be functions defined on D and E respectively, where D and E are subsets of $\mathbb{R}$ satisfying $f(D) \subset E$. Let s be an element of D. Suppose that the function f is continuous at s and that the function g is continuous at $f(s)$. Then the composition $g \circ f$ of f and g is continuous at s.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(u) - g(f(s))| < \varepsilon$ for all $u \in E$ satisfying $|u - f(s)| < \eta$. But then there exists some $\delta > 0$ such that $|f(x) - f(s)| < \eta$ for all $x \in D$ satisfying $|x - s| < \delta$. Thus if $|x - s| < \delta$ then $|g(f(x)) - g(f(s))| < \varepsilon$. Hence $g \circ f$ is continuous at s. ∎

Lemma 3.6 *Let $f\colon D \to \mathbb{R}$ be a function defined on some subset D of $\mathbb{R}$, and let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers belonging to D. Suppose that $a_n \to s$ as $n \to +\infty$, where $s \in D$, and that f is continuous at s. Then $f(a_n) \to f(s)$ as $n \to +\infty$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then there exists some positive integer N such that $|a_n - s| < \delta$ for all n satisfying $n \geq N$. Thus $|f(a_n) - f(s)| < \varepsilon$ for all $n \geq N$. Hence $f(a_n) \to f(s)$ as $n \to +\infty$. ∎

Proposition 3.7 *Let $f\colon D \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$ be functions defined on D and E respectively, where D and E are subsets of $\mathbb{R}$ satisfying $f(D) \subset E$. Let s be a limit point of D, and let l be an element of E. Suppose that $\lim_{x \to s} f(x) = l$ and that the function g is continuous at l. Then $\lim_{x \to s} g(f(x)) = g(l)$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(u) - g(l)| < \varepsilon$ for all $u \in E$ satisfying $|u - l| < \eta$. But then there exists $\delta > 0$ such that $|f(x) - l| < \eta$ for all $x \in D$ satisfying $0 < |x - s| < \delta$. Thus if $0 < |x - s| < \delta$ then $|g(f(x)) - g(l)| < \varepsilon$. Hence $\lim_{x \to s} g(f(x)) = g(l)$. ∎

3.3 The Intermediate Value Theorem

Theorem 3.8 (The Intermediate Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous function defined on the interval $[a, b]$. Let c be a real number which lies between $f(a)$ and $f(b)$ (so that either $f(a) \leq c \leq f(b)$ or else $f(a) \geq c \geq f(b)$.) Then there exists some $s \in [a, b]$ for which $f(s) = c$.*

Proof We first prove the result in the special case in which $c = 0$ and $f(a) \leq 0 \leq f(b)$. We must show that there exists some $s \in [a, b]$ for which $f(s) = 0$. Let S be the subset of $[a, b]$ defined by $S = \{x \in [a, b] : f(x) \leq 0\}$. The set S is non-empty and bounded above (since $a \in S$ and b is an upper bound for the set S). Therefore there exists a least upper bound $\sup S$ for the set S. Let $s = \sup S$. Then $a \leq s \leq b$, since $a \in S$ and $S \subset [a, b]$. We show that $f(s) = 0$.

Now if it were the case that $f(s) \neq 0$. An application of the definition of continuity (with $0 < \varepsilon \leq |f(s)|$) shows that there exists some $\delta > 0$ such that $f(x)$ has the same sign as $f(s)$ for all $x \in [a, b]$ satisfying $|x - s| < \delta$. (Thus if $f(s) > 0$ then $f(x) > 0$ whenever $|x - s| < \delta$, or if $f(s) < 0$ then $f(x) < 0$ whenever $|x - s| < \delta$.)

In particular, suppose that it were the case that $f(s) < 0$. Then $s < b$ (since $f(b) \geq 0$ by hypothesis), and hence $f(x) < 0$, and thus $x \in S$, for some $x \in [a, b]$ satisfying $s < x < s + \delta$. But this would contradict the definition of s.

Next suppose that it were the case that $f(s) > 0$. Then $s > a$ (since $f(a) \leq 0$ by hypothesis), and hence $f(x) > 0$, and thus $x \notin S$, for all $x \in [a, b]$ satisfying $x > s - \delta$. But then $f(x) > 0$ for all $x \geq s - \delta$, and thus $s - \delta$ would be an upper bound of the set S, which also contradicts the definition of s. The only remaining possibility is that $f(s) = 0$, which is what we are seeking to prove.

The result in the general case follows from that in the case $c = 0$ by applying the result in this special case to the function $x \mapsto f(x) - c$ when $f(a) \leq c \leq f(b)$, and to the function $x \mapsto c - f(x)$ when $f(a) \geq c \geq f(b)$. ∎

Corollary 3.9 *Given any positive real number b and natural number n, there exists some positive real number a satisfying $a^n = b$.*

Proof Let $f(x) = x^n - b$, and let $c = \max(b, 1)$. Then $f(0) < 0$ and $f(c) \geq 0$. It follows from the Intermediate Value Theorem (Theorem 3.8) that $f(a) = 0$ for some real number a satisfying $0 < a \leq c$. But then $a^n = b$, as required. ∎

Corollary 3.10 *Let P be a polynomial of odd degree with real coefficients. Then P has at least one real root.*

Proof Let us write $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where n is odd and $a_n \neq 0$. Without loss of generality we may suppose that $a_n > 0$ (after replacing the polynomial P by $-P$ if necessary). We claim that $P(K) > 0$ and $P(-K) < 0$ for some sufficiently large real number K. (Indeed if K is chosen large enough to ensure that $K > 1$ and $|a_n|K \geq 2n|a_j|$ for $j = 0, 1, \ldots, n-1$ then $|P(x) - a_n x^n| \leq \frac{1}{2}|a_n x^n|$ whenever $|x| \geq K$. But then $P(x)$ and $a_n x^n$ have the same sign whenever $|x| \geq K$. In particular, $P(K) > 0$ and $P(-K) < 0$.) It follows immediately from the Intermediate Value Theorem (Theorem 3.8) that there exists some $x_0 \in [-K, K]$ for which $P(x_0) = 0$. Thus the polynomial P has at least one real root. ∎

A function f is said to be *strictly increasing* on an interval I if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$ satisfying $x_1 < x_2$.

The next theorem is useful in verifying the continuity of functions involving square roots, nth roots, inverse trigonometric and logarithm functions.

Theorem 3.11 *A continuous strictly increasing function $f\colon [a, b] \to \mathbb{R}$ on an interval $[a, b]$ has a well-defined continuous inverse $f^{-1}\colon [c, d] \to [a, b]$ defined over the interval $[c, d]$, where $c = f(a)$ and $d = f(b)$.*

Proof It follows from the Intermediate Value Theorem that, for each element y of the interval $[c, d]$, there exists an element x of the interval $[a, b]$ for which $f(x) = y$. There cannot exist two distinct elements u and v of the interval $[a, b]$ satisfying $f(u) = f(v)$, for if $u < v$ then $f(u) < f(v)$, and if $u > v$ then $f(u) > f(v)$. Therefore, for each element y of the interval $[c, d]$, the corresponding element x of the interval $[a, b]$ satisfying $f(x) = y$ is uniquely determined; we denote this element x by $f^{-1}(y)$. We obtain in this fashion a function $f^{-1}\colon [c, d] \to [a, b]$. This function is the inverse function to of $f\colon [a, b] \to [c, d]$.

Let v satisfy $c < v < d$, and let $u = f^{-1}(v)$ (so that $f(u) = v$). We show that f^{-1} is continuous at v. Let $\varepsilon > 0$ be given. Now $a < u < b$, and hence there exist real numbers u_- and u_+ in the interval $[a, b]$ satisfying $u - \varepsilon < u_- < u < u_+ < u + \varepsilon$. Let $v_- = f(u_-)$ and $v_+ = f(u_+)$, and let δ be the minimum of $v_+ - v$ and $v - v_-$. Then $\delta > 0$, and if $y \in [c, d]$ satisfies $|y - v| < \delta$ then $v_- < y < v_+$. But then $u_- < f^{-1}(y) < u_+$, and thus $|f^{-1}(y) - f^{-1}(v)| < \varepsilon$. We deduce that f^{-1} is continuous at v whenever $c < v < d$. A similar proof shows that f^{-1} is continuous at both c and d. ∎

3.4 Continuous Functions on Closed Bounded Intervals

Theorem 3.12 *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function defined on the interval $[a,b]$. Then there exists a constant M with the property that $|f(x)| \leq M$ for all $x \in [a,b]$.*

We give two proofs of this theorem. The first uses the Bolzano-Weierstrass Theorem which states that every bounded sequence of real numbers possesses a convergent subsequence. The second makes use of the Least Upper Bound Axiom.

1st Proof Suppose that the function were not bounded on the interval $[a,b]$. Then there would exist a sequence $x_1, x_2, x_3, \ldots$ of real numbers in the interval $[a,b]$ such that $|f(x_n)| > n$ for all n. The bounded sequence $x_1, x_2, x_3, \ldots$ would possess a convergent subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 2.4). Moreover the limit l of this subsequence would belong to $[a,b]$. But then $f(x_{n_k}) \to f(l)$ as $k \to +\infty$, by Lemma 3.6. It follows that there exists some natural number N with the property that $|f(x_{n_k}) - f(l)| < 1$ whenever $k \geq N$, so that $|f(x_{n_k})| \leq |f(l)| + 1$ whenever $k \geq N$. But this is a contradiction, since $|f(x_{n_k})| > n_k$ for all k and n_k increases without limit as $k \to +\infty$. Thus the function f is indeed bounded on the closed interval $[a,b]$. ∎

2nd Proof Define $S = \{\tau \in [a,b] : f \text{ is bounded on } [a,\tau]\}$. Clearly $a \in S$ and $S \subset [a,b]$. Thus the set S is non-empty and bounded. It follows from the Least Upper Bound axiom that there exists a least upper bound for the set S. Let $s = \sup S$. Then $s \in [a,b]$. The function f is continuous at s. Therefore there exists some $\delta > 0$ such that $|f(x) - f(s)| < 1$, and thus $|f(x)| < |f(s)| + 1$, for all $x \in [a,b]$ satisfying $|x - s| < \delta$.

Now $s - \delta$ is not an upper bound for the set S and hence $s - \delta < \tau \leq s$ for some $\tau \in S$. But then the function f is bounded on $[a,\tau]$ (since $\tau \in S$) and on $[\tau, s]$ (since $|f(s)| + 1$ is an upper bound on this interval). We conclude that f is bounded on $[a,s]$, and thus $s \in S$. Moreover if it were the case that $s < b$ then the function f would be bounded on $[a,x]$, and thus $x \in S$, for all $x \in [a,b]$ satisfying $s < x < s + \delta$, contradicting the definition of s as the least upper bound of the set S. Thus $s = b$. But then $b \in S$, so that the function f is bounded on the interval $[a,b]$ as required. ∎

Theorem 3.13 *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function defined on the interval $[a,b]$. Then there exist $u, v \in [a,b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a,b]$.*

Proof Let $C = \sup\{f(x) : a \leq x \leq b\}$. If there did not exist any $v \in [a, b]$ for which $f(v) = C$ then the function $x \mapsto 1/(C - f(x))$ would be a continuous function on the interval $[a, b]$ which was not bounded above on this interval, thus contradicting Theorem 3.12. Thus there must exist some $v \in [a, b]$ with the property that $f(v) = C$. A similar proof shows that there must exist some $u \in [a, b]$ with the property that $g(u) = c$, where $c = \inf\{f(x) : a \leq x \leq b\}$. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$, as required. ∎

3.5 One-Sided Limits and Limits involving Infinity

Let $f: D \to \mathbb{R}$ be a real-valued function defined on some subset D of $\mathbb{R}$. We write $\lim_{x \to s^+} f(x) = l$ if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for all $x \in D$ satisfying $s < x < s + \delta$. Similarly we write $\lim_{x \to s^-} f(x) = l$ if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for all $x \in D$ satisfying $s - \delta < x < s$. Let $f: D \to \mathbb{R}$ be a real valued function. Note that $\lim_{x \to s^+} f(x) = l$ if and only if $f(x) \to l$ as $x \to s$ in $D \cap \{x \in \mathbb{R} : x > s\}$, and $\lim_{x \to s^-} f(x) = l$ if and only if $f(x) \to l$ as $x \to s$ in $D \cap \{x \in \mathbb{R} : x < s\}$, Also $\lim_{x \to s} f(x) = l$ if and only if both $\lim_{x \to s^+} f(x) = l$ and $\lim_{x \to s^-} f(x) = l$.

We next give the formal definition of the limit of a function of a real variable x as $x \to +\infty$ or $x \to -\infty$.

Let $f: \mathbb{R} \to \mathbb{R}$ be a function of a real variable. We write $\lim_{x \to +\infty} f(x) = l$, where l is some real number, if and only if, given any $\varepsilon > 0$, there exists some real number L with the property that $|f(x) - l| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $x > L$. Similarly we write $\lim_{x \to -\infty} f(x) = l$ if and only if, given any $\varepsilon > 0$, there exists some real number L with the property that $|f(x) - l| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $x < L$.

Example Using the formal definition of the limit, one can show that

$$\lim_{x \to +\infty} \frac{x^2}{1 + x^2} = 1.$$

Indeed suppose that $\varepsilon > 0$ is given. Choose L such that $L > 1/\sqrt{\varepsilon}$. If $x > L$ then

$$\frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2} > 1 - \frac{1}{L^2} > 1 - \varepsilon$$

and thus $\left| \frac{x^2}{1 + x^2} - 1 \right| < \varepsilon$, as required.

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of $\mathbb{R}$, and let s be a limit point of D where $s \notin D$. We write $\lim_{x\to s} f(x) = +\infty$ if, given any real number K (no matter how large), there exists some $\delta > 0$ such that $f(x) > K$ for all $x \in D$ satisfying $0 < |x - s| < \delta$. Similarly we write $\lim_{x\to s} f(x) = -\infty$ if and only if, given any real number K, there exists some $\delta > 0$ such that $f(x) < K$ for all $x \in D$ satisfying $0 < |x - s| < \delta$.

Example One can use the formal definition given above to verify that $\lim_{x\to 0} 1/x^2 = +\infty$. Indeed let the real number K be given. If $K > 0$ choose δ large enough to ensure that $\delta > 1/\sqrt{K}$. Otherwise let δ be any positive real number. If $|x| < \delta$ then $1/x^2 > K$, as required.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function of a real variable. We write $\lim_{x\to+\infty} f(x) = +\infty$ if and only if, given any real number K (no matter how large), there exists a real number L such that $f(x) > K$ for all $x \in \mathbb{R}$ satisfying $x > L$.

In an analogous fashion one can associate a precise meaning to the statements '$\lim_{x\to+\infty} f(x) = -\infty$', '$\lim_{x\to-\infty} f(x) = +\infty$' and '$\lim_{x\to-\infty} f(x) = -\infty$'.