Course 121: Hilary Term 2004

D. R. Wilkins

4 Differentiation

Let s be a real number. We say that a function $f\colon D \to \mathbb{R}$ is defined *around* s if there exists some $\delta > 0$ with the property that all real numbers x satisfying $|x - s| < \delta$ belong to the domain D of the function f.

Definition Let s be some real number, and let f be a real-valued function defined around s. The function f is said to be *differentiable* at s, with *derivative* $f'(s)$, if and only if the limit

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

is well-defined. We denote by f', or by $\dfrac{df}{dx}$ the function whose value at s is the derivative $f'(s)$ of f at s.

Suppose that the real-valued function f is defined around some real number s and is differentiable at s. Then

$$\begin{aligned} \lim_{x \to s} f(x) &= \lim_{h \to 0} f(s+h) = \lim_{h \to 0} f(s) + \left(\lim_{h \to 0} h\right)\left(\lim_{h \to 0} \frac{f(s+h) - f(s)}{h}\right) \\ &= f(s) + 0.f'(s) = f(s), \end{aligned}$$

and therefore f is continuous at s (see Lemma 3.3). Thus differentiability implies continuity.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. Let s be a real number. If $h \neq 0$ then

$$\frac{f(s+h) - f(s)}{h} = \frac{(s+h)^2 - s^2}{h} = 2s + h.$$

Therefore the function f is differentiable at s, and $f'(s) = \lim_{h \to 0}(2s + h) = 2s$.

Example Let $g\colon [0,+\infty) \to \mathbb{R}$ be the function defined by $g(x) = \sqrt{x}$, and let $s \in (0,+\infty)$. If h is any real number satisfying $h > -s$ and $h \neq 0$ then

$$\begin{aligned}\frac{g(s+h)-g(s)}{h} &= \frac{\sqrt{s+h}-\sqrt{s}}{h} = \frac{(\sqrt{s+h}-\sqrt{s})(\sqrt{s+h}+\sqrt{s})}{h(\sqrt{s+h}+\sqrt{s})} \\ &= \frac{(s+h)-s}{h(\sqrt{s+h}+\sqrt{s})} = \frac{1}{\sqrt{s+h}+\sqrt{s}}.\end{aligned}$$

Now $\lim\limits_{h\to 0} \sqrt{s+h} = \sqrt{s}$ (since the function $x \mapsto \sqrt{x}$ is continuous at s). It follows that the function g is differentiable at s, and

$$g'(s) = \lim_{h\to 0} \frac{g(s+h)-g(s)}{h} = \frac{1}{\lim\limits_{h\to 0}(\sqrt{s+h}-\sqrt{s})} = \frac{1}{2\sqrt{s}}.$$

Theorem 4.1 *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s. Then $f+g$ and $f-g$ are differentiable at s, and*

$$(f+g)'(s) = f'(s) + g'(s), \qquad (f-g)'(s) = f'(s) - g'(s).$$

Proof It follows from Proposition 3.2 that

$$\begin{aligned}&\lim_{h\to 0} \frac{(f+g)(s+h)-(f+g)(s)}{h} \\ &\qquad = \lim_{h\to 0} \frac{f(s+h)-f(s)}{h} + \lim_{h\to 0} \frac{g(s+h)-g(s)}{h} \\ &\qquad = f'(s) + g'(s).\end{aligned}$$

Thus the function $f+g$ is differentiable at s, and $(f+g)'(s) = f'(s) + g'(s)$. An analogous proof shows that the function $f-g$ is also differentiable at s and $(f-g)'(s) = f'(s) - g'(s)$. ∎

Proposition 4.2 (Product Rule) *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s. Then $f \cdot g$ is also differentiable at s, and $(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s)$.*

Proof Note that

$$\begin{aligned}&\frac{f(s+h)g(s+h)-f(s)g(s)}{h} \\ &\qquad = \frac{f(s+h)-f(s)}{h} g(s+h) + f(s)\frac{g(s+h)-g(s)}{h}.\end{aligned}$$

Moreover $\lim_{h\to 0} g(s+h) = g(s)$ since g is differentiable, and hence continuous, at s. It follows that

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(s+h)g(s+h) - f(s)g(s)}{h} \\
&\qquad = \lim_{h\to 0} \frac{f(s+h)-f(s)}{h} \lim_{h\to 0} g(s+h) + f(s) \lim_{h\to 0} \frac{g(s+h)-g(s)}{h} \\
&\qquad = f'(s)g(s) + f(s)g'(s).
\end{aligned}$$

Thus the function $f \cdot g$ is differentiable at s, and $(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s)$, as required. ∎

Proposition 4.3 (Quotient Rule) *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s and that the function g is non-zero around s. Then f/g is differentiable at s, and*

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2}.$$

Proof Note that

$$\begin{aligned}
\frac{f(s+h)}{g(s+h)} - \frac{f(s)}{g(s)} &= \frac{f(s+h)g(s) - f(s)g(s+h)}{g(s+h)g(s)} \\
&= \frac{\left(f(s+h)-f(s)\right)g(s) - f(s)\left(g(s+h)-g(s)\right)}{g(s)g(s+h)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(f/g)'(s) &= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(s+h)}{g(s+h)} - \frac{f(s)}{g(s)}\right) \\
&= \frac{1}{g(s)^2}\left(\lim_{h\to 0} \frac{f(s+h)-f(s)}{h} g(s) - f(s) \lim_{h\to 0} \frac{g(s+h)-g(s)}{h}\right) \\
&= \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2},
\end{aligned}$$

since $\lim_{h\to 0} g(s)g(s+h) = g(s)^2 > 0$. ∎

Proposition 4.4 (Chain Rule) *Let a be some real number, let f be a real-valued function defined around a, and let g be a real-valued function defined around $f(a)$. Suppose that the function f is differentiable at a, and the function g is differentiable at $f(a)$. Then the composition function $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Proof Let $b = f(a)$, and let

$$R(y) = \begin{cases} \dfrac{g(y) - g(b)}{y - b} & \text{if } y \neq b; \\ g'(b) & \text{if } y = b. \end{cases}$$

for values of y around b. By considering separately the cases when $f(a+h) \neq f(a)$ and $f(a + h) = f(a)$, we see that

$$g(f(a + h)) - g(f(a)) = R(f(a + h))(f(a + h) - f(a)).$$

Now the function f is continuous at a, because it is differentiable at a. Also the function R is continuous at b, where $b = f(a)$, since

$$\lim_{y \to b} R(y) = \lim_{y \to b} \frac{g(y) - g(b)}{y - b} = \lim_{k \to 0} \frac{g(b + k) - g(b)}{k} = g'(b) = R(b).$$

It follows from Proposition 3.5 that the composition function $R \circ f$ is continuous at a, and therefore

$$\lim_{h \to 0} R(f(a + h)) = R(f(a)) = g'(f(a))$$

by Lemma 3.3. It follows that $g \circ f$ is differentiable at a, and

$$\begin{aligned} (g \circ f)'(a) &= \lim_{h \to 0} \frac{g(f(a + h)) - g(f(a))}{h} \\ &= \lim_{h \to 0} R(f(a + h)) \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = g'(f(a))f'(a), \end{aligned}$$

as required. ▮

4.1 Rolle's Theorem and the Mean Value Theorem

Theorem 4.5 (Rolle's Theorem) *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.*

Proof First we show that if the function f attains its minimum value at u, and if $a < u < b$, then $f'(u) = 0$. Now the difference quotient

$$\frac{f(u + h) - f(u)}{h}$$

is non-negative for all sufficiently small positive values of h; therefore $f'(u) \geq 0$. On the other hand, this difference quotient is non-positive for all sufficiently small negative values of h; therefore $f'(u) \leq 0$. We deduce therefore that $f'(u) = 0$.

Similarly if the function f attains its maximum value at v, and if $a < v < b$, then $f'(v) = 0$. (Indeed the result for local maxima can be deduced from the corresponding result for local minima simply by replacing the function f by $-f$.)

Now the function f is continuous on the closed bounded interval $[a, b]$. It therefore follows from Theorem 3.13 that there must exist real numbers u and v in the interval $[a, b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$. If $a < u < b$ then $f'(u) = 0$ and we can take $s = u$. Similarly if $a < v < b$ then $f'(v) = 0$ and we can take $s = v$. The only remaining case to consider is when both u and v are endpoints of $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 4.6 (The Mean Value Theorem) *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$f(b) - f(a) = f'(s)(b - a).$$

Proof Let $g\colon [a, b] \to \mathbb{R}$ be the real-valued function on the closed interval $[a, b]$ defined by

$$g(x) = f(x) - \frac{b - x}{b - a} f(a) - \frac{x - a}{b - a} f(b).$$

Then the function g is continuous on $[a, b]$ and differentiable on (a, b). Moreover $g(a) = 0$ and $g(b) = 0$. It follows from Rolle's Theorem (Theorem 4.5) that $g'(s) = 0$ for some real number s satisfying $a < s < b$. But

$$g'(s) = f'(s) - \frac{f(b) - f(a)}{b - a}.$$

Therefore $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

We now prove a generalization of the standard Mean Value Theorem, known as *Cauchy's Mean Value Theorem*.

Theorem 4.7 (Cauchy's Mean Value Theorem) *Let f and g be real-valued functions defined on some interval $[a, b]$. Suppose that f and g are continuous on $[a, b]$ and are differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$(f(b) - f(a))\, g'(s) = (g(b) - g(a))\, f'(s).$$

In particular, if $g(b) \neq g(a)$ and the function g' is non-zero throughout (a, b), then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(s)}{g'(s)}.$$

Proof Consider the function $h\colon [a, b] \to \mathbb{R}$ defined by

$$h(x) = f(x)\,(g(b) - g(a)) - g(x)\,(f(b) - f(a))\,.$$

Then $h(a) = f(a)g(b) - g(a)f(b) = h(b)$, and the function h satisfies the hypotheses of Rolle's Theorem on the interval $[a, b]$. We deduce from Rolle's Theorem (Theorem 4.5) that $h'(s) = 0$ for some s satisfying $a < s < b$. The required result then follows immediately. ∎

An important corollary of Cauchy's Mean Value Theorem is *l'Hôpital's Rule* for evaluating the limit of a quotient of two functions at a point where both functions vanish.

Corollary 4.8 (L'Hôpital's Rule) *Let f and g be differentiable real-valued functions defined around some real number s for which $f(s) = g(s) = 0$. Suppose that there exists $\delta > 0$ such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $0 < |x - s| < \delta$, and that the limit of $f'(x)/g'(x)$ exists (and is finite) as $x \to s$. Then the limit of $f(x)/g(x)$ exists as $x \to s$, and*

$$\lim_{x \to s} \frac{f(x)}{g(x)} = \lim_{x \to s} \frac{f'(x)}{g'(x)}.$$

Proof Let l be the limit of $f'(x)/g'(x)$ as $x \to s$, and let $\varepsilon > 0$ be given. By choosing $\delta > 0$ sufficiently small we can ensure that $f(x)/g(x)$ and $f'(x)/g'(x)$ are well-defined and

$$\left| \frac{f'(x)}{g'(x)} - l \right| < \varepsilon$$

for all x satisfying $0 < |x - s| < \delta$. Moreover, by applying Cauchy's Mean Value Theorem to the functions f and g on the interval with endpoints s and x (considering separately the cases $x > s$ and $x < s$), we deduce that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(s)}{g(x) - g(s)} = \frac{f'(u)}{g'(u)}.$$

for some real number u between s and x. (Recall that $f(s) = g(s) = 0$.) Thus if $0 < |x - s| < \delta$ then $0 < |u - s| < \delta$, and hence

$$\left|\frac{f(x)}{g(x)} - l\right| = \left|\frac{f'(u)}{g'(u)} - l\right| < \varepsilon.$$

This shows that $f(x)/g(x) \to l$ as $x \to s$, as required. ∎

Example Using l'Hôpital's Rule twice, we see that

$$\lim_{x\to 2} \frac{x^3 + x^2 - 16x + 20}{x^3 - 3x^2 + 4} = \lim_{x\to 2} \frac{3x^2 + 2x - 16}{3x^2 - 6x} = \lim_{x\to 2} \frac{6x + 2}{6x - 6} = \frac{7}{3}.$$

4.2 Taylor's Theorem

An *open interval* in $\mathbb{R}$ is an interval I with the property that, given any $s \in I$, there exists some $\delta > 0$ such that every real number x satisfying $|x - s| < \delta$ belongs to the interval I. Given real numbers c and d satisfying $c < d$, the intervals (c, d), $(c, +\infty)$ and $(-\infty, d)$ are open intervals, as is the whole real line $\mathbb{R}$.

Theorem 4.9 (Taylor's Theorem) *Let s and h be real numbers, and let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$. Then*

$$f(s + h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h)$$

for some real number θ satisfying $0 < \theta < 1$.

Proof Let $p\colon [0, 1] \to \mathbb{R}$ be defined by

$$p(t) = f(s + th) - f(s) - \sum_{n=1}^{k-1} \frac{t^n h^n}{n!} f^{(n)}(s).$$

A straightforward calculation shows that $p^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$. Thus if $q(t) = p(t) - p(1)t^k$ for all $s \in [0, 1]$ then $q^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$, and $q(1) = 0$. We can therefore apply Rolle's Theorem (Theorem 4.5) to the function q on the interval $[0, 1]$ to deduce the existence of some real number t_1 satisfying $0 < t_1 < 1$ for which $q'(t_1) = 0$. We can then apply Rolle's Theorem to the function q' on the interval $[0, t_1]$ to deduce the existence of some real number t_2 satisfying $0 < t_2 < t_1$ for which $q''(t_2) = 0$.

By continuing in this fashion, applying Rolle's Theorem in turn to the functions $q'', q''', \ldots, q^{(k-1)}$, we deduce the existence of real numbers $t_1, t_2, \ldots, t_k$ satisfying $0 < t_k < t_{k-1} < \cdots < t_1 < 1$ with the property that $q^{(n)}(t_n) = 0$ for $n = 1, 2, \ldots, k$. Let $\theta = t_k$. Then $0 < \theta < 1$ and

$$0 = \frac{1}{k!} q^{(k)}(\theta) = \frac{1}{k!} p^{(k)}(\theta) - p(1) = \frac{h^k}{k!} f^{(k)}(s + \theta h) - p(1),$$

hence

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + p(1) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h),$$

as required. ∎

Example Consider the cosine function $\cos\colon \mathbb{R} \to \mathbb{R}$. The derivatives of this function are given by $\cos^{(2n)}(x) = (-1)^n \cos(x)$ and $\cos^{(2n-1)}(x) = (-1)^n \sin(x)$ for all natural numbers n. It follows from Taylor's Theorem that, given any $x \in \mathbb{R}$ and given any non-negative integer m, there exists some θ satisfying $0 < \theta < 1$ such that

$$\cos x = \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} + \frac{x^{2m+1}(-1)^{m-1}}{(2m+1)!} \sin(\theta x)$$

(The value of θ will depend on x and m.) But then

$$\left| \cos x - \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} \right| \leq \frac{|x|^{2m+1}}{(2m+1)!},$$

hence

$$\cos x = \lim_{m \to +\infty} \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

A similar argument shows that

$$\sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Let f be an infinitely differentiable real-valued function defined around some real number a. The infinite series

$$f(a) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(a)$$

is referred to as the *Taylor expansion* of the function f about a. It will often be the case that

$$f(a+h) = f(a) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(a) = f(a) + \lim_{m \to +\infty} \left(\sum_{n=1}^{m} \frac{h^n}{n!} f^{(n)}(a) \right)$$

for all sufficiently small values of h. However there exist functions whose Taylor expansion about some real number a does not converge to the given function for any non-zero value of h. Such a function is exhibited in the following example.

Example Let

$$g(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

We show by induction on k that the function g is k times differentiable on $\mathbb{R}$ and $g^{(k)}(0) = 0$ for all natural numbers k. Now

$$g^{(k)}(x) = \frac{p_k(x)}{x^{2k}} \exp(-(1/x))$$

for all $x > 0$, where $p_1(x) = 1$ and

$$p_{k+1}(x) = x^2 p_k'(x) + (1 - 2kx) p_k(x)$$

for all k. Note that $p_1, p_2, p_3, \ldots$ are polynomials in x. Also elementary calculus shows that the function $x \mapsto x^{-2k-1} \exp(-1/x)$ attains its maximum value C_k on $(0, +\infty)$ when $(2k+1)x = 1$, and therefore $0 < x^{-2k} \exp(-1/x) \leq C_k x$ for all $x > 0$. It follows that $\lim\limits_{x \to 0^+} x^{-2k} \exp(-1/x) = 0$, and hence

$$g^{(k)}(0) = \lim_{x \to 0} \frac{g^{(k-1)}(x)}{x} = 0 = \lim_{x \to 0} g^{(k)}(x)$$

for all natural numbers k, as required.

Note that the function g has a well-defined Taylor expansion about $x = 0$. Moreover all the terms of this Taylor expansion are zero, and therefore the Taylor expansion of g converges to the zero function. This function therefore provides an example of a function where the Taylor expansion is well-defined but does not converge to the given function.

5 Integration

A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *lower sum* $L(P, f)$ and the *upper sum* $U(P, f)$ of f for the partition P of $[a, b]$ are defined by

$$L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}), \qquad U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$. Clearly $L(P, f) \le U(P, f)$. Moreover $\sum_{i=1}^{n} (x_i - x_{i-1}) = b - a$, and therefore

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a),$$

for any real numbers m and M satisfying $m \le f(x) \le M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ of the function f on $[a, b]$ are defined by

$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &\equiv \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}, \\ \mathcal{L}\int_a^b f(x)\,dx &\equiv \sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \end{aligned}$$

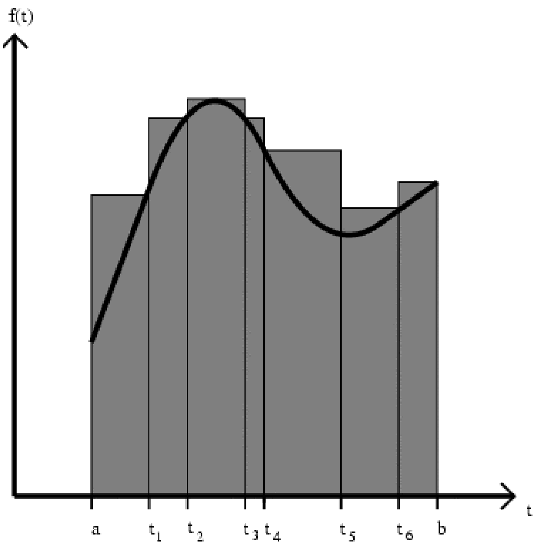
(i.e., $\mathcal{U}\int_a^b f(x)\,dx$ is the infimum of the values of $U(P, f)$ and $\mathcal{L}\int_a^b f(x)\,dx$ is the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$). If

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx$$

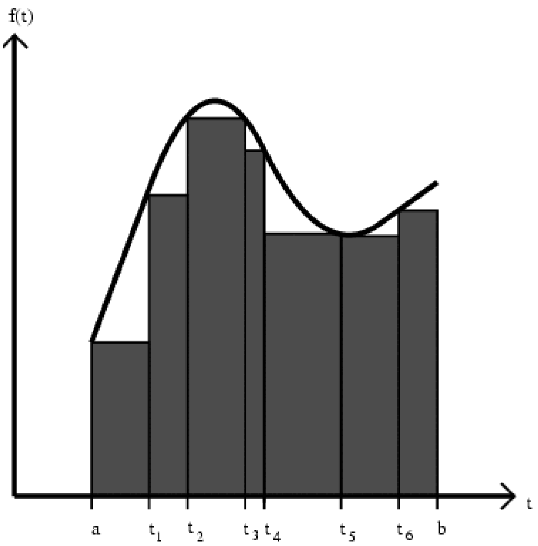
then the function f is said to be *Riemann-integrable* on $[a, b]$, and the *Riemann integral* $\int_a^b f(x)\,dx$ of f on $[a, b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$



The upper sum $U(P, f)$



The lower sum $L(P, f)$

for all Riemann-integrable functions f on $[b, a]$. We set $\int_a^b f(x) = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$ for all partitions P of $[a, b]$.

Definition Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 5.1 *Let R be a refinement of some partition P of $[a, b]$. Then $L(R, f) \geq L(P, f)$ and $U(R, f) \leq U(P, f)$ for any bounded function $f\colon [a, b] \to \mathbb{R}$.*

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$ given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^{n} L(R_i, f)$ and $U(R, f) = \sum_{i=1}^{n} U(R_i, f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 5.2 *Let f be a bounded real-valued function on the interval $[a, b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 5.1 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality

$L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each natural number n, let P_n denote the partition of $[0, 1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Thus

$$\begin{aligned} L(P_n, f) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d - \frac{c}{n}\right) \\ &= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\ U(P_n, f) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d\right) \\ &= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}. \end{aligned}$$

But $L(P_n, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P_n, f)$ for all n. It follows that $\mathcal{L}\int_a^b f(x)\,dx = \frac{1}{2}c + d = \mathcal{U}\int_a^b f(x)\,dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\,dx = \frac{1}{2}c + d$.

Example Let $f\colon [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0, 1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0, 1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0, 1]$.

It can be shown that sums and products of Riemann-integrable functions are themselves Riemann-integrable.

Proposition 5.3 *Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let Q and R be any partitions of the intervals $[a, b]$ and $[b, c]$ respectively. These partitions combine to give a partition $Q \cup R$ of the interval $[a, c]$: thus if $Q = \{a, x_1, \ldots, x_{n-1}, b\}$ and $R = \{b, u_1, \ldots, u_{m-1}, c\}$, where

$$a < x_1 < x_2 < \cdots < x_{n-1} < b < u_1 < u_2 < \cdots < u_{m-1} < c,$$

then $Q \cup R = \{a, x_1, \ldots, x_{n-1}, b, u_1, \ldots, u_{m-1}, c\}$. Clearly the lower and upper sums of f satisfy $L(Q, f) + L(R, f) = L(Q \cup R, f)$ and $U(Q, f) + U(R, f) = U(Q \cup R, f)$. It follows that

$$L(Q, f) + L(R, f) \leq \mathcal{L}\int_a^c f(x)\,dx.$$

Taking the supremum of the left hand side of this inequality over all partitions Q of $[a, b]$ and all partitions R of $[b, c]$, we deduce that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

Similarly $U(Q, f) + U(R, f) \geq \mathcal{U}\int_a^c f(x)\,dx$, and hence

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \geq \mathcal{U}\int_a^c f(x)\,dx.$$

But $\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx$ by Lemma 5.2. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx,$$

as required. ∎

5.1 Integrability of Continuous functions

Lemma 5.4 *Let f be a continuous real-valued function on a closed bounded interval $[a,b]$. Then, given any $\varepsilon > 0$, there exists some $\delta > 0$ (independent of x and y) such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in [a,b]$ satisfying $|x - y| < \delta$.*

Proof Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ with the required property. Then, for each natural number n, there would exist values x_n and y_n in the interval $[a,b]$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. But the sequence $x_1, x_2, x_3, \ldots$ would be bounded (since $a \leq x_n \leq b$ for all n) and thus would possess a convergent subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 2.4). Let $l = \lim_{k \to +\infty} x_{n_k}$. Then $l = \lim_{k \to +\infty} y_{n_k}$ also, since $\lim_{k \to +\infty} (y_{n_k} - x_{n_k}) = 0$. Moreover $a \leq l \leq b$, and therefore

$$\lim_{k \to +\infty} f(x_{n_k}) = \lim_{k \to +\infty} f(y_{n_k}) = f(l),$$

since f is continuous (see Lemma 3.6). Thus $\lim_{k \to +\infty} (f(x_{n_k}) - f(y_{n_k})) = 0$. But this is impossible, since x_n and y_n have been chosen so that $|f(x_n) - f(y_n)| \geq \varepsilon$ for all n. We conclude therefore that there must exist some $\delta > 0$ with the required property. ∎

Theorem 5.5 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a,b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a,b]$. Then f is bounded above and below on the interval $[a,b]$ (see Theorem 3.12).

Let $\varepsilon > 0$ be given. It follows from Lemma 5.4 that there exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a,b]$ satisfy $|x - y| < \delta$. Choose a partition P of the interval $[a,b]$ such that each subinterval in the partition has length less than δ. Write $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Now if $x_{i-1} \leq x \leq x_i$ then $|x - x_i| < \delta$, and hence $f(x_i) - \varepsilon < f(x) < f(x_i) + \varepsilon$. It follows that

$$f(x_i) - \varepsilon \leq m_i \leq M_i \leq f(x_i) + \varepsilon \qquad (i = 1, 2, \ldots, n),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Therefore

$$\begin{aligned} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \varepsilon(b - a) &\leq L(P, f) \leq U(P, f) \\ &\leq \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \varepsilon(b - a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P, and hence

$$0 \leq \mathcal{U} \int_a^b f(x)\,dx - \mathcal{L} \int_a^b f(x)\,dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b - a).$$

But this inequality must be satisfied for any ε satisfying $\varepsilon > 0$. Therefore

$$\mathcal{U} \int_a^b f(x)\,dx = \mathcal{L} \int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

5.2 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 5.5). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 5.6 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a, b)$. Now the function f is continuous at x, where $a < x < b$. Thus, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(x)| < \frac{1}{2}\varepsilon$ for all $t \in [a, b]$ satisfying $|t - x| < \delta$. Now

$$\frac{F(x + h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt.$$

But if $0 < |h| < \delta$ and $x + h \in [a, b]$ then $\left|\int_x^{x+h} (f(t) - f(x))\,dt\right| \leq \frac{1}{2}\varepsilon|h|$, and thus

$$\left|\frac{F(x + h) - F(x)}{h} - f(x)\right| \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

It follows that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \lim_{h\to 0} \frac{F(x + h) - F(x)}{h} = f(x),$$

as required. ∎

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. We say that f is *continuously differentiable* on $[a,b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \qquad f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

exist at the endpoints of $[a,b]$, and the function f' is continuous on $[a,b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h \to 0^+} \frac{F(a+h) - F(a)}{h} = f(a), \qquad \lim_{h \to 0^-} \frac{F(b+h) - F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Corollary 5.7 *Let f be a continuously differentiable real-valued function on the interval $[a,b]$. Then*

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)$$

Proof Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\,dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\,dt\right) = 0$$

for all x satisfing $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 4.6) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b-a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

Corollary 5.8 (Integration by Parts) *Let f and g be continuously differentiable real-valued functions on the interval $[a,b]$. Then*

$$\int_a^b f(t)\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b \frac{df(x)}{dx}g(x)\,dx.$$

Proof This result follows from Corollary 5.7 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - \frac{df(x)}{dx}g(x). \quad \blacksquare$$

Corollary 5.9 (Integration by Substitution) *Let $u\colon [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then*

$$\int_c^d f(x)\,dx = \int_a^b f(u(t))\frac{du(t)}{dt}\,dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof Let F and G be the functions on $[a,b]$ defined by

$$F(x) = \int_c^{u(x)} f(y)dy, \qquad G(x) = \int_a^x f(u(t))\frac{du(t)}{dt}dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where $H(s) = \int_c^s f(y)\,dy$, and $H'(s) = f(s)$ for all $s \in [a,b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a,b)$. On applying the Mean Value Theorem (Theorem 4.6) to the function $F-G$ on the interval $[a,b]$, we see that $F(b)-G(b) = F(a)-G(a) = 0$. Thus $F(b) = G(b) = H(d)$, which yields the required identity. $\blacksquare$

5.3 Interchanging Limits and Integrals, Uniform Convergence

Let $f_1, f_2, f_3, \ldots$ be a sequence of Riemann-integrable functions defined over the interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Suppose that the sequence $f_1(x), f_2(x), f_3(x)$ converges for all $x \in [a,b]$. We wish to determine whether or not

$$\lim_{n\to+\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to+\infty} f_n(x)\right)\,dx.$$

The following example demonstrates that this identity can fail to hold, even when the functions involved are well-behaved polynomial functions.

Example Let $f_1, f_2, f_3, \ldots$ be the sequence of continuous functions on the interval $[0,1]$ defined by $f_n(x) = n(x^n - x^{2n})$. Now

$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to+\infty} \left(\frac{n}{n+1} - \frac{n}{2n+1} \right) = \frac{1}{2}.$$

On the other hand, we shall show that $\lim\limits_{n\to+\infty} f_n(x) = 0$ for all $x \in [0,1]$. Thus one cannot interchange limits and integrals in this case.

Suppose that $0 \leq x < 1$. We claim that $nx^n \to 0$ as $n \to +\infty$. To verify this, choose u satisfying $x < u < 1$. Then $0 \leq (n+1)u^{n+1} \leq nu^n$ for all n satisfying $n > u/(1-u)$. Therefore there exists some constant B with the property that $0 \leq nu^n \leq B$ for all n. But then $0 \leq nx^n \leq B(x/u)^n$ for all n, and $(x/u)^n \to 0$ as $n \to +\infty$. Therefore $nx^n \to 0$ as $n \to +\infty$, as claimed. It follows that

$$\lim_{n\to+\infty} f_n(x) = \left(\lim_{n\to+\infty} nx^n \right) \left(\lim_{n\to+\infty} (1 - x^n) \right) = 0$$

for all x satisfying $0 \leq x < 1$. Also $f_n(1) = 0$ for all n. We conclude that $\lim\limits_{n\to+\infty} f_n(x) = 0$ for all $x \in [0,1]$, which is what we set out to show.

We now introduce the concept of *uniform convergence*. Later shall show that, given a sequence $f_1, f_2, f_3, \ldots$ of Riemann-integrable functions on some interval $[a,b]$, the identity

$$\lim_{n\to+\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to+\infty} f_n(x) \right)\,dx.$$

is valid, provided that the sequence $f_1, f_2, f_3, \ldots$ of functions converges *uniformly* on the interval $[a,b]$.

Definition Let $f_1, f_2, f_3, \ldots$ be a sequence of real-valued functions defined on some subset D of $\mathbb{R}$. The sequence (f_n) is said to converge *uniformly* to a function f on D as $n \to +\infty$ if and only if the following criterion is satisfied:

for every $\varepsilon > 0$, there exists some positive integer N such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in D$ and for all integers n satisfying $n \geq N$ (where the value of N is independent of x).

Let $f_1, f_2, f_3, \ldots$ be a sequence of bounded real-valued functions on some subset D of $\mathbb{R}$ which converges uniformly on D to the zero function. For each natural number n, let $M_n = \sup\{f_n(x) : x \in D\}$. We claim that $M_n \to 0$ as $n \to +\infty$. To prove this, let $\varepsilon > 0$ be given. Then there exists some natural number N such that $|f_n(x)| < \frac{1}{2}\varepsilon$ for all $x \in D$ and $n \geq N$. Thus if $n \geq N$ then $M_n \leq \frac{1}{2}\varepsilon < \varepsilon$. This shows that $M_n \to 0$ as $n \to +\infty$, as claimed.

Example Let $(f_n : n \in \mathbb{N})$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_n(x) = n(x^n - x^{2n})$. We have already shown (in an earlier example) that $\lim_{n\to+\infty} f_n(x) = 0$ for all $x \in [0, 1]$. However a straightforward exercise in Calculus shows that the maximum value attained by the function f_n is $n/4$ (which is attained at $x = 1/2^{\frac{1}{n}}$), and $n/4 \to +\infty$ as $n \to +\infty$. It follows from this that the sequence $f_1, f_2, f_3, \dots$ does not converge uniformly to the zero function on the interval $[0, 1]$.

Proposition 5.10 *Let $f_1, f_2, f_3, \dots$ be a sequence of continuous real-valued functions defined on some subset D of $\mathbb{R}$. Suppose that this sequence converges uniformly on D to some real-valued function f. Then f is continuous on D.*

Proof Let s be an element of D, and let $\varepsilon > 0$ be given. If n is chosen sufficiently large then $|f(x) - f_n(x)| < \frac{1}{3}\varepsilon$ for all $x \in D$, since $f_n \to f$ uniformly on D as $n \to +\infty$. It then follows from the continuity of f_n that there exists some $\delta > 0$ such that $|f_n(x) - f_n(s)| < \frac{1}{3}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$|f(x)-f(s)| \leq |f(x)-f_n(x)|+|f_n(x)-f_n(s)|+|f_n(s)-f(s)| < \tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon = \varepsilon$$

whenever $|x - s| < \delta$. Thus the function f is continuous at s, as required. ∎

Theorem 5.11 *Let $f_1, f_2, f_3, \dots$ be a sequence of continuous real-valued functions which converges uniformly on the interval $[a, b]$ to some continuous real-valued function f. Then*

$$\lim_{n\to+\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof Let $\varepsilon > 0$ be given. Choose ε_0 small enough to ensure that $0 < \varepsilon_0(b - a) < \varepsilon$. Then there exists some natural number N such that $|f_n(x) - f(x)| < \varepsilon_0$ for all $x \in [a, b]$ and $n \geq N$, since the sequence $f_1, f_2, f_3, \dots$ of functions converges uniformly to f on $[a, b]$. Now

$$-\int_a^b |f_n(x) - f(x)|\,dx \leq \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \leq \int_a^b |f_n(x) - f(x)|\,dx.$$

It follows that

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx \leq \varepsilon_0(b - a) < \varepsilon,$$

whenever $n \geq N$. The result follows. ∎

5.4 Integrals over Unbounded Intervals

We define integrals over unbounded intervals by appropriate limiting processes. Given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $[a, +\infty)$, we define

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. Similarly, given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. If f is bounded and Riemann integrable over each closed bounded interval in $\mathbb{R}$ then we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a \to -\infty, b \to +\infty} \int_a^b f(x)\,dx,$$

provided that this limit exists.

6 Analysis in the Complex Plane

A *complex number* is a number of the form $x+iy$, where x and y are real numbers and $i^2 = -1$. Arithmetic operations on the set $\mathbb{C}$ of complex numbers are defined as follows:

$$\begin{aligned}
(x+iy)+(u+iv) &= (x+u)+i(y+v),\\
(x+iy)-(u+iv) &= (x-u)+i(y-v),\\
(x+iy)(u+iv) &= (xu-yv)+i(xv+yu),\\
(x+iy)/(u+iv) &= \frac{xu+yv}{u^2+v^2}+i\frac{yu-xv}{u^2+v^2} \qquad (u+iv \neq 0).
\end{aligned}$$

If $z \in \mathbb{C}$ is given by $z = x + iy$, where x and y are real numbers, then the *real part* $\operatorname{Re} z$, *imaginary part* $\operatorname{Im} z$, *complex conjugate* $\overline{z}$ and *modulus* $|z|$ of z are given by

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y, \quad \overline{z} = x - iy, \quad |z| = +\sqrt{x^2 + y^2}.$$

Note that $\overline{(z+w)} = \overline{z} + \overline{w}$, $\overline{(zw)} = \overline{z}\,\overline{w}$, $|-z| = |z|$, $|\overline{z}| = |z|$ and $|z|^2 = z\overline{z}$ for all complex numbers z and w. Moreover $|z| = 0$ if and only if $z = 0$.

Lemma 6.1 *$|zw| = |z||w|$ and $|z + w| \le |z| + |w|$ for all $z, w \in \mathbb{C}$.*

Proof Let us write $z = x + iy$ and $w = u + iv$, where $x, y, u, v \in \mathbb{R}$. Then

$$\begin{aligned} |zw|^2 &= |xu - yv + i(xv + yu)|^2 = (xu - yv)^2 + (xv + yu)^2 \\ &= (x^2u^2 + y^2v^2 - 2xuyv) + (x^2v^2 + y^2u^2 + 2xvyu) \\ &= (x^2 + y^2)(u^2 + v^2) = |z|^2|w|^2, \end{aligned}$$

so that $|zw| = |z||w|$. Moreover

$$\begin{aligned} |z + w|^2 &= (x + u)^2 + (y + v)^2 = x^2 + u^2 + y^2 + v^2 + 2xu + 2yv \\ &= |z|^2 + |w|^2 + 2(xu + yv). \end{aligned}$$

But

$$(xu + yv) \le +\sqrt{(xu + yv)^2 + (yu - xv)^2} = |z\overline{w}|$$

and $|z\overline{w}| = |z||\overline{w}| = |z||w|$. Thus

$$|z + w|^2 \le |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2,$$

so that $|z + w| \le |z| + |w|$, as required. ∎

6.1 Infinite Sequences of Complex Numbers

Definition A sequence $z_1, z_2, z_3, \ldots$ of complex numbers is said to *converge* to some complex number l if the following criterion is satisfied:

> for every real number ε satisfying $\varepsilon > 0$ there exists some natural number N such that $|z_n - l| < \varepsilon$ for all natural numbers n satisfying $n \ge N$.

The complex number l is referred to as the *limit* of the sequence $z_1, z_2, z_3, \ldots$, and is denoted by $\lim_{n\to+\infty} z_n$.

A sequence $z_1, z_2, z_3, \ldots$ of complex numbers is said to be *bounded* if there exists some real number $R \ge 0$ such that $|z_n| \le R$ for all n. Every convergent sequence of complex numbers is bounded.

Example The sequence $w, w^2, w^3, \ldots$ convergens to 0 for any complex number w satisfying $|w| < 1$. Indeed suppose that $\varepsilon > 0$ is given. If N is chosen such that $N > \log\varepsilon / \log|w|$ then $|w^n| < \varepsilon$ for all $n \ge N$.

Lemma 6.2 *Let $z_1, z_2, z_3, \ldots$ be an infinite sequence of complex numbers, and, for each n, let $z_n = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$. Then $\lim_{n \to +\infty} z_n = l$ for some $l \in \mathbb{C}$ if and only if $\lim_{n \to +\infty} x_n = \lambda$ and $\lim_{n \to +\infty} y_n = \mu$, where $\lambda + i\mu = l$.*

Proof Suppose that $z_n \to l$ as $n \to +\infty$, where $l = \lambda + i\mu$. Then, given any $\varepsilon > 0$, there exists some natural number N such that $|z_n - l| < \varepsilon$ whenever $n \geq N$. But then $|x_n - \lambda| < \varepsilon$ and $|y_n - \mu| < \varepsilon$ whenever $n \geq N$. We conclude that $x_n \to \lambda$ and $y_n \to \mu$ as $n \to +\infty$.

Conversely suppose that $x_n \to \lambda$ and $y_n \to \mu$ as $n \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist natural numbers N_1 and N_2 such that $|x_n - \lambda| < \varepsilon/\sqrt{2}$ whenever $n \geq N_1$ and $|y_n - \mu| < \varepsilon/\sqrt{2}$ whenever $n \geq N_2$. Let N be the maximum of N_1 and N_2. If $n \geq N$ then

$$|z_n - l|^2 = |x_n - \lambda|^2 + |y_n - \mu|^2 < \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{2}\varepsilon^2 = \varepsilon^2,$$

and thus $|z_n - l| < \varepsilon$. This shows that $z_n \to l$ as $n \to +\infty$, as required. ∎

Proposition 6.3 *Let (z_n) and (w_n) be convergent infinite sequences of complex numbers. Then the sequences $(z_n + w_n)$, $(z_n - w_n)$ and $(z_n w_n)$ are convergent, and*

$$\begin{aligned}
\lim_{n \to +\infty} (z_n + w_n) &= \lim_{n \to +\infty} z_n + \lim_{n \to +\infty} w_n, \\
\lim_{n \to +\infty} (z_n - w_n) &= \lim_{n \to +\infty} z_n - \lim_{n \to +\infty} w_n, \\
\lim_{n \to +\infty} (z_n w_n) &= \left(\lim_{n \to +\infty} z_n\right)\left(\lim_{n \to +\infty} w_n\right).
\end{aligned}$$

If in addition $w_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to +\infty} w_n \neq 0$, then the sequence (z_n/w_n) is convergent, and

$$\lim_{n \to +\infty} \frac{z_n}{w_n} = \frac{\lim_{n \to +\infty} z_n}{\lim_{n \to +\infty} w_n}.$$

Proof This result can be proved either by a straightforward adaptation of the proof of Proposition 2.2, or else by splitting the sequences into their real and imaginary parts, and using Lemma 6.2 and Proposition 2.2. ∎

A *subsequence* of a given sequence $z_1, z_2, z_3, \ldots$ of complex numbers is a sequence of the form $z_{n_1}, z_{n_2}, z_{n_3}, \ldots$, where $n_1 < n_2 < n_3 < \cdots$.

Theorem 6.4 (Bolzano-Weierstrass) *Every bounded sequence of complex numbers has a convergent subsequence*

Proof Let $z_1, z_2, z_3, \ldots$ be a bounded sequence of complex numbers, and let $z_n = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$. The Bolzano-Weierstrass Theorem for sequences of real numbers (Theorem 2.4) guarantees the existence of a subsequence $z_{n_1}, z_{n_2}, z_{n_3}, \ldots$ of the given sequence such that the real parts $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ converge. A further application of Theorem 2.4 then allows to replace this subsequence by a subsequence of itself in order to ensure that the imaginary parts $y_{n_1}, y_{n_2}, y_{n_3}, \ldots$ also converge. But then $z_{n_1}, z_{n_2}, z_{n_3}, \ldots$ is a convergent subsequence of $z_1, z_2, z_3, \ldots$, by Lemma 6.2. ∎

6.2 Cauchy's Criterion for Convergence

Definition A sequence $z_1, z_2, z_3, \ldots$ of complex numbers is said to be a *Cauchy sequence* if the following condition is satisfied:

> for every real number ε satisfying $\varepsilon > 0$ there exists some natural number N such that $|z_m - z_n| < \varepsilon$ for all natural numbers m and n satisfying $m \geq N$ and $n \geq N$.

Lemma 6.5 *Every Cauchy sequence of complex numbers is bounded.*

Proof Let $z_1, z_2, z_3, \ldots$ be a Cauchy sequence. Then there exists some natural number N such that $|z_n - z_m| < 1$ whenever $m \geq N$ and $n \geq N$. In particular, $|z_n| \leq |z_N| + 1$ whenever $n \geq N$. Therefore $|z_n| \leq R$ for all n, where R is the maximum of the real numbers $|z_1|, |z_2|, \ldots, |z_{N-1}|$ and $|z_N|+1$. Thus the sequence is bounded, as required. ∎

The following important result is known as *Cauchy's Criterion for convergence*, or as the *General Principle of Convergence*.

Theorem 6.6 (Cauchy's Criterion for Convergence) *An infinite sequence of complex numbers is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences are Cauchy sequences. Let $z_1, z_2, z_3, \ldots$ be a convergent sequence, and let $l = \lim\limits_{n \to +\infty} z_j$. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $|z_n - l| < \frac{1}{2}\varepsilon$ for all $n \geq N$. Thus if $m \geq N$ and $n \geq N$ then $|z_m - l| < \frac{1}{2}\varepsilon$ and $|z_n - l| < \frac{1}{2}\varepsilon$, and hence

$$|z_m - z_n| = |(z_m - l) - (z_n - l)| \leq |z_m - l| + |z_n - l| < \varepsilon.$$

Thus the sequence $z_1, z_2, z_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $z_1, z_2, z_3, \ldots$ is convergent. Now Cauchy sequences are bounded, by Lemma 6.5. The sequence $z_1, z_2, z_3, \ldots$ therefore has a convergent subsequence $z_{n_1}, z_{n_2}, z_{n_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 6.4). Let $l = \lim_{j \to +\infty} z_{n_j}$. We claim that the sequence $z_1, z_2, z_3, \ldots$ itself converges to l.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $|z_n - z_m| < \frac{1}{2}\varepsilon$ whenever $m \geq N$ and $n \geq N$ (since the sequence is a Cauchy sequence). Let j be chosen large enough to ensure that $n_j \geq N$ and $|z_{n_j} - l| < \frac{1}{2}\varepsilon$. Then

$$|z_n - l| \leq |z_n - z_{n_j}| + |z_{n_j} - l| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $n \geq N$, and thus $z_n \to l$ as $n \to +\infty$, as required. ∎

6.3 Limits of Functions of a Complex Variable

Let D be a subset of the set $\mathbb{C}$ of complex numbers A complex number w is said to be a *limit point* of D if and only if, given any $\delta > 0$, there exists $z \in D$ satisfying $0 < |z - w| < \delta$.

Definition Let $f: D \to \mathbb{C}$ be a function defined over some subset D of $\mathbb{C}$. Let w be a limit point of D. A complex number l is said to be the *limit* of the function f as z tends to w in D if, given any real number ε satisfying $\varepsilon > 0$, there exists some real number δ satisfying $\delta > 0$ such that $|f(z) - l| < \varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta$.

If l is the limit of $f(z)$ as z approaches some limit point w of the domain of the function f then we denote this fact either by writing '$f(z) \to l$ as $z \to w$' or by writing '$\lim\limits_{z \to w} f(z) = l$'. A straightforward adaptation of the proof of Lemma 3.1 shows that the limit $\lim\limits_{z \to w} f(z)$, if it exists, is unique.

Proposition 6.7 *Let $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ be functions defined over some subset D of $\mathbb{C}$. Let w be a limit point of D. Suppose that $\lim\limits_{z \to w} f(z)$ and $\lim\limits_{z \to w} g(z)$ exist. Then $\lim\limits_{z \to w} (f(z) + g(z))$, $\lim\limits_{z \to w} (f(z) - g(z))$ and $\lim\limits_{z \to w} (f(z)g(z))$ exist, and*

$$\begin{aligned} \lim_{z \to w} (f(z) + g(z)) &= \lim_{z \to w} f(z) + \lim_{z \to w} g(z), \\ \lim_{z \to w} (f(z) - g(z)) &= \lim_{z \to w} f(z) - \lim_{z \to w} g(z), \\ \lim_{z \to w} (f(z)g(z)) &= \lim_{z \to w} f(z) \lim_{z \to w} g(z). \end{aligned}$$

If in addition $g(z) \neq 0$ *for all* $z \in D$ *and* $\lim_{z\to w} g(z) \neq 0$, *then* $\lim_{z\to w} f(z)/g(z)$ *exists, and*

$$\lim_{z\to w} \frac{f(z)}{g(z)} = \frac{\lim_{z\to w} f(z)}{\lim_{z\to w} g(z)}.$$

Proof Let $l = \lim_{z\to w} f(z)$ and $m = \lim_{z\to w} g(z)$.

First we prove that $\lim_{z\to w}(f(z) + g(z)) = l + m$. Let $\varepsilon > 0$ be given. We must prove that there exists some $\delta > 0$ such that $|f(z)+g(z)-(l+m)| < \varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta$. Now there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(z) - l| < \frac{1}{2}\varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta_1$, and $|g(z) - m| < \frac{1}{2}\varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta_2$, since $l = \lim_{z\to w} f(z)$ and $m = \lim_{z\to w} g(z)$. Let δ be the minimum of δ_1 and δ_2. If $z \in D$ satisfies $0 < |z - w| < \delta$ then $|f(z) - l| < \frac{1}{2}\varepsilon$ and $|g(z) - m| < \frac{1}{2}\varepsilon$, and hence

$$|f(z) + g(z) - (l + m)| \leq |f(z) - l| + |g(z) - m| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

This shows that $\lim_{z\to w}(f(z) + g(z)) = l + m$.

Let c be some complex number. We show that $\lim_{z\to w}(cg(z)) = cm$. The case when $c = 0$ is trivial. Suppose that $c \neq 0$. Let $\varepsilon > 0$ be given. Then there exists some real number $\delta > 0$ such that $|g(z) - m| < \varepsilon/|c|$ whenever $0 < |z - w| < \delta$. But then $|cg(z) - cm| = |c||g(z) - m| < \varepsilon$ whenever $0 < |z - w| < \delta$. Thus $\lim_{z\to w}(cg(z)) = cm$.

If we combine this result, for $c = -1$, with the previous result, we see that $\lim_{z\to w}(-g(z)) = -m$, and therefore $\lim_{z\to w}(f(z) - g(z)) = l - m$.

Next we show that if $p\colon D \to \mathbb{R}$ and $q\colon D \to \mathbb{R}$ are functions with the property that $\lim_{z\to w} p(z) = \lim_{z\to w} q(z) = 0$, then $\lim_{z\to w}(p(z)q(z)) = 0$. Let $\varepsilon > 0$ be given. Then there exist real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|p(z)| < \sqrt{\varepsilon}$ whenever $0 < |z - w| < \delta_1$ and $|q(z)| < \sqrt{\varepsilon}$ whenever $0 < |z - w| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $0 < |z - w| < \delta$ then $|p(z)q(z)| < \varepsilon$. We deduce that $\lim_{z\to w}(p(z)q(z)) = 0$.

We can apply this result with $p(z) = f(z) - l$ and $q(z) = g(z) - m$ for all $z \in D$. Using the results we have already obtained, we see that

$$\begin{aligned} 0 &= \lim_{z\to w}(p(z)q(z)) = \lim_{z\to w}(f(z)g(z) - f(z)m - lg(z) + lm) \\ &= \lim_{z\to w}(f(z)g(z)) - m\lim_{z\to w} f(z) - l\lim_{z\to w} g(z) + lm = \lim_{z\to w}(f(z)g(z)) - lm. \end{aligned}$$

Thus $\lim_{z\to w}(f(z)g(z)) = lm$.

Next we show that if $h: D \to R$ is a function that is non-zero throughout D, and if $\lim_{z \to w} h(z) \to 1$ then $\lim_{z \to w} (1/h(z)) = 1$. Let $\varepsilon > 0$ be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some $\delta > 0$ such that $|h(z) - 1| < \varepsilon_0$ whenever $0 < |z - w| < \delta$. Thus if $0 < |z - w| < \delta$ then $|h(z) - 1| < \frac{1}{2}\varepsilon$ and $|h(z)| \geq 1 - |1 - h(z)| > \frac{1}{2}$. But then

$$\left|\frac{1}{h(z)} - 1\right| = \left|\frac{h(z) - 1}{h(z)}\right| = \frac{|h(z) - 1|}{|h(z)|} < 2|h(z) - 1| < \varepsilon.$$

We deduce that $\lim_{z \to w} 1/h(z) = 1$. If we apply this result with $h(z) = g(z)/m$, where $m \neq 0$, we deduce that $\lim_{z \to w} m/g(z) = 1$, and thus $\lim_{z \to w} 1/g(z) = 1/m$. The result we have already obtained for products of functions then enables us to deduce that $\lim_{z \to w} (f(z)/g(z)) \to l/m$.

The proof of Proposition 6.7 is exactly analogous to the proof of the corresponding result for real-valued functions (Proposition 3.2).

6.4 Continuous Functions of a Complex Variable.

Let D be a subset of $\mathbb{C}$, and let $f: D \to \mathbb{C}$ be a function on D. Let w be a point of D. The function f is said to be *continuous* at w if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for all $z \in D$ satisfying $|z - w| < \delta$. If f is continuous at every point of D then we say that f is continuous on D.

Let $f: D \to \mathbb{C}$ be a function defined on some subset D of $\mathbb{C}$, and let $w \in D$. Suppose that w is a limit point of D. On comparing definitions, we see that the function f is continuous at w if and only if $\lim_{z \to w} f(z) = f(w)$. If $w \in D$ is not a limit point of D then any function defined on D is continuous there.

Given functions $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ defined over some subset D of $\mathbb{C}$, we denote by $f + g$, $f - g$, $f \cdot g$ and f/g the functions on D defined by

$$(f + g)(z) = f(z) + g(z), \qquad (f - g)(z) = f(z) - g(z),$$

$$(f \cdot g)(z) = f(z)g(z), \qquad (f/g)(z) = f(z)/g(z).$$

Proposition 6.8 *Let $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ be functions defined over some subset D of $\mathbb{C}$. Suppose that f and g are continuous at some point w of D. Then the functions $f + g$, $f - g$ and $f \cdot g$ are also continuous at w. If moreover the function g is everywhere non-zero on D then the function f/g is continuous at w.*

Proof This result follows directly using Proposition 6.7 and the relationship between continuity and limits described above. ∎

Proposition 6.9 *Let $f\colon D \to \mathbb{C}$ and $g\colon E \to \mathbb{C}$ be functions defined on D and E respectively, where D and E are subsets of $\mathbb{C}$ satisfying $f(D) \subset E$. Let w be an element of D. Suppose that the function f is continuous at w and that the function g is continuous at $f(w)$. Then the composition $g \circ f$ of f and g is continuous at w.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\zeta) - g(f(w))| < \varepsilon$ for all $\zeta \in E$ satisfying $|\zeta - f(w)| < \eta$. But then there exists some $\delta > 0$ such that $|f(z) - f(w)| < \eta$ for all $z \in D$ satisfying $|z - w| < \delta$. Thus if $|z - w| < \delta$ then $|g(f(z)) - g(f(w))| < \varepsilon$. Hence $g \circ f$ is continuous at w. ∎

Lemma 6.10 *Let $f\colon D \to \mathbb{C}$ be a function defined on some subset D of $\mathbb{C}$, and let $z_1, z_2, z_3, \ldots$ be a sequence of complex numbers belonging to D. Suppose that $z_n \to w$ as $n \to +\infty$, where $w \in D$, and that f is continuous at w. Then $f(z_n) \to f(w)$ as $n \to +\infty$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for all $z \in D$ satisfying $|z - w| < \delta$. But then there exists some positive integer N such that $|z_n - w| < \delta$ for all n satisfying $n \geq N$. Thus $|f(z_n) - f(w)| < \varepsilon$ for all $n \geq N$. Hence $f(z_n) \to f(w)$ as $n \to +\infty$. ∎

Proposition 6.11 *Let $f\colon D \to \mathbb{C}$ and $g\colon E \to \mathbb{C}$ be functions defined on D and E respectively, where D and E are subsets of $\mathbb{C}$ satisfying $f(D) \subset E$. Let w be a limit point of D, and let l be an element of E. Suppose that $\lim_{z \to w} f(z) = l$ and that the function g is continuous at l. Then $\lim_{z \to w} g(f(z)) = g(l)$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\zeta) - g(l)| < \varepsilon$ for all $\zeta \in E$ satisfying $|\zeta - l| < \eta$. But then there exists $\delta > 0$ such that $|f(z) - l| < \eta$ for all $z \in D$ satisfying $0 < |z - w| < \delta$. Thus if $0 < |z - w| < \delta$ then $|g(f(z)) - g(l)| < \varepsilon$. Hence $\lim_{z \to w} g(f(z)) = g(l)$. ∎

6.5 Uniform Convergence

Let D be a subset of $\mathbb{C}$ and let $f_1, f_2, f_3, \ldots,$ be a sequence of functions mapping D into $\mathbb{C}$. We say that the infinite sequence $f_1, f_2, f_3, \ldots$ converges *uniformly* on D to a function $f\colon D \to \mathbb{C}$ if, given any $\varepsilon > 0$, there exists some natural number N such that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in D$ and for all natural numbers n satisfying $n \geq N$, where the value of N chosen does not depend on the value of z.

Theorem 6.12 *Let D be a subset of $\mathbb{C}$, and let $f_1, f_2, f_3, \ldots$ be a sequence of continuous functions mapping D into $\mathbb{C}$ which is uniformly convergent on D to some function $f\colon D \to \mathbb{C}$. Then the function f is continuous on D.*

Proof Let w be an element of D. We wish to show that the function f is continuous at w. Let $\varepsilon > 0$ be given. We must show that there exists some $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $z \in D$ satisfies $|z - w| < \delta$. Now we can find some value of N, independent of z, with the property that $|f_n(z) - f(z)| < \frac{1}{3}\varepsilon$ for all $z \in D$ and for all $n \geq N$. Choose any n satisfying $n \geq N$. We can find some $\delta > 0$ such that $|f_n(z) - f_n(w)| < \frac{1}{3}\varepsilon$ whenever $z \in D$ satisfies $|z - w| < \delta$, since the function f_n is continuous at w. But then

$$\begin{aligned} |f(z) - f(w)| &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(w)| + |f_n(w) - f(w)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

whenever $|z-w| < \delta$. Thus the function f is continuous at w, as required. ∎