Course 121, 1993–94, Test I (JF Michaelmas Term)

Answer Question 1 and TWO other questions

1. (a) Let D be a set of real numbers. Define what is meant by saying that the set D is *bounded above*, and what is meant by saying that some real number s is the *least upper bound* (or *supremum*) of the set D. State the *Least Upper Bound Axiom*.

   (b) Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real numbers, and let l be a real number. Define precisely what is meant by saying that the sequence $a_1, a_2, a_3, \ldots$ *converges* to l.

   (c) Determine whether or not each of the following limits exists, and, if so, what is the value of the limit:—

   $$\lim_{n\to+\infty} \frac{2n^2+n}{6n^4+5}, \qquad \lim_{n\to+\infty} \frac{6n^3-7n}{2n^3+2n}, \qquad \lim_{n\to+\infty} \frac{n+3}{\sqrt{n}}.$$

   (d) Let (u_n) and (v_n) be infinite sequences of real numbers. Suppose that $u_n \to 0$ and $v_n \to 0$ as $n \to +\infty$. Prove that $u_n v_n \to 0$ as $n \to +\infty$.

2. State and prove the *Bolzano-Weierstrass Theorem*

3. Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be infinite sequences of real numbers. Suppose that $\lim_{n\to+\infty} b_n = 0$ and that there exists some positive real number K such that $|a_n| \leq K$ for all natural numbers n. Use the definition of convergence to prove that $\lim_{n\to+\infty} (a_n b_n) = 0$.

4. Let (a_n), (b_n) and (c_n) be infinite sequences of real numbers. Suppose that $a_n \leq b_n \leq c_n$ for all natural numbers n and $\lim_{n\to+\infty} (c_n - b_n) = 0$. Prove that the sequence (b_n) is convergent if and only if the sequence (a_n) is convergent, in which case $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n$.

Course 121, 1993–94, Test II (JF Michaelmas Term)

Answer Question 1 and TWO other questions

1. (a) Let $f: D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s be a limit point of D. Let l be a real number. Define precisely what is meant by saying that l is the *limit* of $f(x)$ as x tends to s in D.

   (b) Define precisely what it means to say that a real-valued function $f: D \to \mathbb{R}$ defined over some subset D of $\mathbb{R}$ is *continuous* at some real number s belonging to D.

   (c) Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be functions defined over some subset D of $\mathbb{R}$, and let s be a limit point of D. Suppose that $\lim_{x \to s} f(x)$ and $\lim_{x \to s} g(x)$ exist. Prove that $\lim_{x \to s}(f(x) + g(x))$ exists, and

   $$\lim_{x \to s}(f(x) + g(x)) = \lim_{x \to s} f(x) + \lim_{x \to s} g(x).$$

   (d) Let $f: D \to \mathbb{R}$ be a continuous function defined over some subset D of $\mathbb{R}$, and let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers belonging to D which converges to some element l of D. Prove that the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(l)$.

   (e) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} x^3 \cos\left(\dfrac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

   Is the function f continuous at 0? [Fully justify your answer.]

2. State and prove the *Intermediate Value Theorem.*

3. (a) Let a and b be continuous real numbers satisfying $a < b$ and let $f: [a, b] \to \mathbb{R}$ be a continuous function defined on the interval $[a, b]$, where $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. Prove that there exists some positive real number M such that $|f(x)| \leq M$ for all $x \in [a, b]$.

   (b) Let $f: [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(x) > 0$ for all $x \in [a, b]$. Prove that there exists some strictly positive real number c such that $f(x) \geq c > 0$ for all $x \in [a, b]$.

4. Let $f: D \to \mathbb{R}$ be a function defined over some subset D of $\mathbb{R}$, and let s be a limit point of D. Suppose that $f(x) \geq 0$ for all $x \in D$ and that $f(x) \to l$ as $x \to s$ in D. Prove that $l \geq 0$.

Course 121, 1993–94, Test III (JF Hilary Term)

Answer Question 1 and TWO other questions

1. (a) Let a be a real number, and let f be a real-valued function defined around a. What is meant by saying that the function f is *differentiable* at a.

   (b) State and prove *Rolle's Theorem*.

   (c) Let D be a subset of $\mathbb{R}$. What is meant by saying that a sequence $f_1, f_2, f_3, \ldots$ of real valued functions from D to $\mathbb{R}$ converges *uniformly* on D to a limit function $f\colon D \to \mathbb{R}$.

   (d) Let a and b be real numbers satisfying $a < b$, and let $f_1, f_2, f_3, \ldots$ be a sequence of continuous functions which converges uniformly on the interval $[a, b]$ to some function $f\colon [a, b] \to \mathbb{R}$. Prove that the function f is continuous on $[a, b]$.

2. Let p, q and r be real numbers. Prove that the cubic polynomial $x^3 + px^2 + qx + r$ has exactly one real root if its coefficients satisfy the inequality $p^2 < 3q$.

3. State and prove *Taylor's Theorem*.

4. (a) State and prove the *Fundamental Theorem of Calculus*.

   (b) Let a and b be real numbers satisfying $a < b$, and let $f_1, f_2, f_3, \ldots$ be a sequence of continuous functions from $[a, b]$ to $\mathbb{R}$ that converge uniformly on the interval $[a, b]$ to a continuous function $f\colon [a, b] \to \mathbb{R}$. Prove that $\int_a^b f(x)\,dt = \lim_{n \to +\infty} \int_a^b f_n(x)\,dx$.

5. (a) What is meant by saying that a bounded subset S of the Euclidean plane is *Jordan-measurable*.

   (b) Prove that one can assign to each Jordan-measurable set S in the Euclidean plane a unique real number $\mathrm{area}(S)$ characterized by the following properties: if P is a polygonal region and $P \subset S$ then $\mathrm{area}(P) \leq \mathrm{area}(S)$, and if Q is a polygonal region and $S \subset Q$ then $\mathrm{area}(S) \leq \mathrm{area}(Q)$. [You may use, without proof, the fact that $\mathrm{area}(P) \leq \mathrm{area}(Q)$ whenever P and Q are polygonal regions in the plane satisfying $P \subset Q$.]

Course 121, 1993–94, Test IV (JF Hilary Term)

Answer Question 1 and TWO other questions

1. (a) What is meant by saying that a infinite sequence $z_1, z_2, z_3, \ldots$ of complex numbers is *convergent*?

   (b) Let $l = \lambda + i\mu$, where λ and μ are real numbers and $i = \sqrt{-1}$, and let $z_n = x_n + iy_n$ for all natural numbers n, where (x_n) and (y_n) are infinite sequences of real numbers. Prove that that the infinite sequence (z_n) converges to l if and only if the infinite sequences (x_n) and (y_n) converge to λ and μ respectively.

   (c) Test the following infinite series for convergence:—

$$\sum_{n=1}^{+\infty} \frac{\sqrt{n}+1}{2n\sqrt{n}-1}, \qquad \sum_{n=1}^{+\infty} \frac{3\cos n - 2}{n^3}, \qquad \sum_{n=1}^{+\infty} \frac{1}{(2n)!}.$$

2. (a) What is meant by saying that a infinite sequence $z_1, z_2, z_3, \ldots$ of complex numbers is a *Cauchy sequence*?

   (b) Prove that any Cauchy sequence of complex numbers is bounded.

   (c) Prove that an infinite sequence of complex numbers is convergent if and only if it is a Cauchy sequence. (This is *Cauchy's Criterion for convergence*, also known as the *General Principle of Convergence*).

3. Prove that the infinite series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is convergent.

4. (a) What is meant by saying that an infinite series $\sum_{n=1}^{+\infty} a_n$ of complex numbers is *absolutely convergent*?

   (b) Let D be a subset of $\mathbb{C}$ and let $f_1, f_2, f_3, \ldots$ be functions from D to $\mathbb{C}$. What is meant by saying that the infinite series $\sum_{n=1}^{+\infty} f_n(z)$ of functions is *uniformly convergent* on D?

   (c) Let $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ be absolutely convergent series of complex numbers. Prove that if

$$f(x) = \sum_{n=1}^{+\infty} \bigl(a_n \cos(nx) + b_n \sin(nx)\bigr)$$

   then the infinite series defining $f(x)$ converges for all real numbers x. Prove also that the function $f\colon \mathbb{R} \to \mathbb{C}$ is continuous on $\mathbb{R}$. [You may assume, without proof, any general results concerning uniformly convergent infinite series that you require.]

Course 121, 1993–94, Test V (JF Trinity Term)

Answer Question 1 and TWO other questions

1. (a) What is meant by saying that a subset V of $\mathbb{R}^n$ is an *open set* in $\mathbb{R}^n$? What is meant by saying that a subset F of $\mathbb{R}^n$ is a *closed set* in $\mathbb{R}^n$?

   (b) Let $\mathbf{p}$ be a point of $\mathbb{R}^n$. Prove that the open ball $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$ of radius r about the point $\mathbf{p}$ is an open set in $\mathbb{R}^n$.

   (c) Let c be a real number. Prove that the set $\{(x, y) \in \mathbb{R}^2 : y > c\}$ is an open set in $\mathbb{R}^2$.

   (d) Consider the following subsets of $\mathbb{R}^2$. Determine whether or not they are open, and also whether or not they are closed in $\mathbb{R}^2$. [Fully justify your answers.]

   (i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 9 \text{ or } y \leq 1\}$,
   (ii) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 9 \text{ or } y \leq 1\}$,

2. (a) Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. What is meant by saying that a function $f\colon X \to Y$ from X to Y is *continuous*?

   (b) Prove that the function $s\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ is continuous.

   (c) Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Prove that a function $f\colon X \to Y$ is continuous if and only if its components are continuous.

   (d) Using (b) and (c), or otherwise, show that if $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous functions on a subset X of $\mathbb{R}^m$ then so is $f + g$, where $(f + g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$ for all $\mathbf{x} \in X$.

3. Prove that the collection of open sets in $\mathbb{R}^n$ has the following properties:—

   (i) the empty set $\emptyset$ and $\mathbb{R}^n$ itself are open sets;

   (ii) any union of open sets is itself an open set;

   (iii) any finite intersection of open sets is an open set.

4. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function, let K be a closed bounded set in $\mathbb{R}^m$, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. Suppose that $V \cap f(K)$ is non-empty for all open sets V that contain the point $\mathbf{q}$. Prove that there exists $\mathbf{p} \in K$ for which $f(\mathbf{p}) = \mathbf{q}$. [You may use, without proof, any result proved in the lecture notes, provided that the result is clearly stated.]

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