Course 121, 1993–94, Supplemental Examination (JF)

1. Prove that there do not exist non-zero integers p and q satisfying $p^2 = 2q^2$.

2. (a) Define precisely what is meant by saying that an infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers *converges* to some real number l.

   (b) Prove that a non-decreasing sequence of real numbers is convergent if it is bounded above.

3. (a) Let $D \subset \mathbb{R}$ and let $s \in D$. Define precisely what is meant by saying that a function $f\colon D \to \mathbb{R}$ is *continuous* at s.

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

   $$f(x) = \begin{cases} x^2(1 - x^3)\sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

   Using the formal definition of continuity, prove that the function f is continuous at 0.

   (c) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers which converges to some real number l. Suppose that $a_n \geq 0$ for all n. Prove that $l \geq 0$.

4. Let $[a, b]$ be a closed bounded interval. Prove that every continuous function from $[a, b]$ to $\mathbb{R}$ is bounded. Hence show that, given any continuous function $f\colon [a, b] \to \mathbb{R}$, there exist elements u and v of $[a, b]$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$.

5. State and prove *Taylor's Theorem* (with remainder).

6. (a) State and prove the *Fundamental Theorem of Calculus.*

   (b) Let f be a real-valued function that is continuously differentiable on some open interval containing $[a, b]$ where $a < b$. Using the Fundamental Theorem of Calculus and the Mean Value Theorem, or otherwise, show that

   $$\int_a^b \frac{df(x)}{dx}\, dx = f(b) - f(a).$$

   (c) Use the result stated in (b) to derive the rule for Integration by Parts.

7. (a) Suppose that $z_n = x_n + iy_n$ for all n, where x_n and y_n are real numbers. Prove that the sequence (z_n) converges to $\lambda + i\mu$, where λ and μ are real numbers, if and only if the sequence (x_n) converges to λ and the sequence (y_n) converges to μ.

   (b) Let $f\colon D \to \mathbb{C}$ be a continuous function defined over D, where $D \subset \mathbb{C}$, and let $z_1, z_2, z_3, \ldots$ be a sequence of elements of D which converges to some element w of D. Suppose that the function f is continuous at w. Prove that the sequence $f(z_1), f(z_2), f(z_3), \ldots$ converges to $f(w)$.

8. (a) What is a *Cauchy sequence* of complex numbers?

   (b) Prove that every Cauchy sequence of complex numbers is bounded, and hence prove that every Cauchy sequence of complex numbers is convergent. [You may use, without proof, the Bolzano-Weierstrass Theorem for infinite sequences of complex numbers.]

9. Test the following infinite series for convergence:

$$(i) \quad \sum_{n=1}^{+\infty} \frac{2\cos n - 3}{n^3}, \qquad (ii) \quad \sum_{n=1}^{+\infty} \frac{n}{2n^2 - 1},$$

$$(iii) \quad \sum_{n=1}^{+\infty} \frac{n^3}{n!}, \qquad (iv) \quad \sum_{n=1}^{+\infty} \frac{n!}{n^{2n}}.$$

10. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ is convergent for all $\alpha > 1$.

11. (a) Prove that a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to some point $\mathbf{p}$ if and only if, given any open set U containing the point $\mathbf{p}$, there exists some natural number N such that $\mathbf{x}_j \in U$ for all $j \geq N$.

   (b) Suppose that $\mathbf{x}_j \in F$ for all j, where F is a closed set in $\mathbb{R}^n$, and that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to some point $\mathbf{p}$. Prove that $\mathbf{p} \in F$.

12. Consider the following subsets of $\mathbb{R}^3$. Determine for each subset whether or not it is open, and whether or not it is closed in $\mathbb{R}^3$. [Justify your answers.]

   (i) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 16\}$,

(ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9 \text{ and } z > 1\}$,

(iii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 9 \text{ and } z < 1\}$.