# Course 121, 1992–93, Test IV (JF Hilary Term)

## Answer Question 1 and TWO other questions

1. (a) What is meant by saying that a infinite sequence $z_1, z_2, z_3, \ldots$ of complex numbers is *convergent*?

   (b) Let $f\colon D \to \mathbb{C}$ and $g\colon D \to \mathbb{C}$ be functions defined over some subset $D$ of $\mathbb{C}$, and let $w$ be a limit point of $D$. Suppose that $\lim\limits_{z \to w} f(z) = l$ and $\lim\limits_{z \to w} g(z) = m$ for some complex numbers $l$ and $m$. Prove that $\lim\limits_{z \to w} (f(z) + g(z)) = l + m$.

   (c) Test the following infinite series for convergence:—

$$\sum_{n=1}^{+\infty} \frac{(-2)^n + 1}{4^n + 3}, \qquad \sum_{n=1}^{+\infty} \frac{n^2}{2n^3 - 1},$$

$$\sum_{n=1}^{+\infty} \frac{n^2}{n!}, \qquad \sum_{n=1}^{+\infty} \frac{n!}{(2n)^n}.$$

2. Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions from $D$ to $\mathbb{C}$, where $D \subset \mathbb{C}$.

   What is meant by saying that the sequence $f_1, f_2, f_3, \ldots$ converges *uniformly* on $D$ to some function $f\colon D \to \mathbb{C}$.

   (b) Suppose that each function $f_n$ is continuous on $D$ and the sequence $f_1, f_2, f_3, \ldots$ converges uniformly on $D$ to some function $f$. Prove that the limit function $f$ is continuous on $D$.

   (c) What is meant by saying that an infinite series $\sum\limits_{n=1}^{+\infty} f_n(z)$ of functions is *uniformly convergent* on a subset $D$ of $\mathbb{C}$.

   (d) Describe the *Weierstrass M-test* for uniform convergence, and prove the validity of this test. [You may assume the validity of the *Comparison Test* for convergence.]

3. Prove that the series $\sum\limits_{n=1}^{+\infty} \frac{1}{n}$ is divergent.

4. Let $a_1, a_2, a_3, \ldots$ be complex numbers. Suppose that $\lim\limits_{n \to +\infty} |a_n|^{1/n} < 1$.

   Prove that the infinite series $\sum\limits_{n=1}^{+\infty} a_n$ is convergent.

# Course 121, 1992–93, Test V (JF Trinity Term)

## Answer Question 1 and TWO other questions

1. (a) What is meant by saying that a subset $V$ of $\mathbb{R}^n$ is an *open set* in $\mathbb{R}^n$? What is meant by saying that a subset $F$ of $\mathbb{R}^n$ is a *closed set* in $\mathbb{R}^n$?

   (b) Let $\mathbf{p}$ be a point of $\mathbb{R}^n$. Prove that the open ball

   $$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$$

   of radius $r$ about the point $\mathbf{p}$ is an open set in $\mathbb{R}^n$.

   (c) Let $c$ be a real number. Prove that the set $\{(x, y, z) \in \mathbb{R}^3 : z < c\}$ is an open set in $\mathbb{R}^3$.

   (d) Consider the following subsets of $\mathbb{R}^3$. Determine whether or not they are open, and also whether or not they are closed in $\mathbb{R}^3$. [Fully justify your answers.]

   (i) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 4 \text{ or } z \geq 0\}$,
   (ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 4 \text{ or } z < 0\}$,

2. (a) What is meant by saying that a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of $\mathbb{R}^n$ *converges* to some point $\mathbf{p}$ of $\mathbb{R}^n$.

   (b) Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of $\mathbb{R}^n$, and let $x_{ji}$ denote the $i$th component of $\mathbf{x}_j$ for each natural number $j$. Prove that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to some point $\mathbf{p}$ if and only if $\lim_{j \to +\infty} x_{ji} = p_i$ for $i = 1, 2, \ldots, n$, where $p_i$ is the $i$th component of $\mathbf{p}$.

3. Let $X$ and $Y$ be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f: X \to Y$ be a function from $X$ to $Y$.

   (a) What is meant by saying that the function $f$ is *continuous* at some point $\mathbf{p}$ of $X$?

   (b) Prove that $f: X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in $X$ for all subsets $V$ of $Y$ that are open in $Y$.

4. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Suppose that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{x} \neq \mathbf{0}$, and that there exists some $\alpha > 0$ such that $f(\lambda\mathbf{x}) = \lambda^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\lambda > 0$. Prove that there exists some constant $c$ satisfying $c > 0$ such that $f(\mathbf{x}) \geq c|\mathbf{x}|^\alpha$ for all $\mathbf{x} \in \mathbb{R}^n$. [You may use, without proof, any result proved in the lecture notes, provided that the result is clearly stated.]