# Course 121, 1991–92, Test I (JF Michaelmas Term)

## Answer Question 1 and TWO other questions

1. (a) Let $D$ be a set of real numbers. Define what is meant by saying that the set $D$ is *bounded above*, and define precisely what is meant by saying that some real number $c$ is the *least upper bound* (or *supremum*) of the set $D$. State the *Least Upper Bound Axiom*.

   (b) Let $t_1, t_2, t_3, \ldots$ be an infinite sequence of real numbers, and let $l$ be a real number. Define precisely what is meant by saying that the sequence $t_1, t_2, t_3, \ldots$ *converges* to $l$.

   (c) Let $(s_n)$ and $(t_n)$ be convergent sequences of real numbers. Prove that the sequence $(s_n + t_n)$ is convergent, and that

$$\lim_{n \to +\infty} (s_n + t_n) = \lim_{n \to +\infty} s_n + \lim_{n \to +\infty} t_n.$$

   (d) Let $t_1, t_2, t_3, \ldots$ be a non-decreasing sequence of real numbers which is bounded above. Prove that this sequence converges to some real number $l$.

2. (a) Determine whether or not each of the following limits exists, and, if so, what is the value of the limit:—

$$\lim_{n \to +\infty} \frac{n+7}{3n^2+5}, \qquad \lim_{n \to +\infty} \frac{2n^2+7}{3n^2+5}, \qquad \lim_{n \to +\infty} \frac{4n^3+7}{3n^2+5}.$$

   (b) Using the precise definition of convergence, prove that

$$\lim_{n \to +\infty} \frac{1}{\sqrt{n}} \sin(\tfrac{1}{2}\pi n) = 0.$$

3. Let $(s_n)$ and $(t_n)$ be infinite sequences of real numbers. Suppose that

   (i) the sequence $(s_n)$ is non-decreasing,

   (ii) the sequence $(t_n)$ is bounded above,

   (iii) the sequence $(s_n - t_n)$ is convergent.

   Prove that the sequences $(s_n)$ and $(t_n)$ are both convergent.

4. (a) State and prove the *Bolzano-Weierstrass Theorem*

   (b) Let $t_1, t_2, t_3, \ldots$ be a sequence of real numbers which has at least one convergent subsequence. Is this infinite sequence necessarily bounded? [Justify your answer with a proof or counterexample.]

# Course 121, 1991–92, Test II (JF Michaelmas Term)

## Attempt THREE questions Answer Question 1 and TWO other questions

1. (a) Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over some subset $D$ of $\mathbb{R}$, and let $s$ be a limit point of $D$. Let $l$ be a real number. Define precisely what is meant by saying that $l$ is the *limit* of $f(t)$ as $t$ tends to $s$ in $D$.

   (b) Define precisely what it means to say that a real-valued function $f\colon D \to \mathbb{R}$ defined over some subset $D$ of $\mathbb{R}$ is *continuous* at some real number $s$ belonging to $D$.

   (c) Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions defined over some subset $D$ of $\mathbb{R}$, and let $s$ be a limit point of $D$. Suppose that $\lim_{t\to s} f(t)$ and $\lim_{t\to s} g(t)$ exist. Prove that $\lim_{t\to s}(f(t) + g(t))$ exists, and

$$\lim_{t\to s}(f(t) + g(t)) = \lim_{t\to s} f(t) + \lim_{t\to s} g(t).$$

   (d) Let $f\colon D \to \mathbb{R}$ be a continuous function defined over some subset $D$ of $\mathbb{R}$, and let $t_1, t_2, t_3, \ldots$ be a sequence of real numbers belonging to $D$ which converges to some element $l$ of $D$. Prove that the sequence $f(t_1), f(t_2), f(t_3), \ldots$ converges to $f(l)$.

   (e) State the *Intermediate Value Theorem*. [You are not asked to prove the theorem.]

2. (a) Using the definition of the limit of a real-valued function, prove formally that

$$\lim_{t\to 0}\left(\sqrt{t}\cos\left(\frac{1}{t^2}\right)\right) = 0.$$

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} (t - t^2)\sin\left(\dfrac{1}{t^2}\right) & \text{if } t \neq 0; \\ 0 & \text{if } t = 0. \end{cases}$$

   Is the function $f$ continuous at 0? [Justify your answer.]

   (c) Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} \dfrac{1}{1+t^2}\sin\left(\dfrac{1}{t^2}\right) & \text{if } t \neq 0; \\ 0 & \text{if } t = 0. \end{cases}$$

   Is the function $g$ continuous at 0? [Justify your answer.]

3. Let $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a function defined on the set $\mathbb{R}\setminus\{0\}$ of all non-zero real numbers. Suppose that $f(x) \geq f(y)$ for all non-zero real numbers $x$ and $y$ satisfying $|x| \leq |y|$. Suppose also that the function $f$ is bounded above on $\mathbb{R}\setminus\{0\}$ (i.e., there exists a constant $B$ such that $f(t) \leq B$ for all non-zero real numbers $t$). Prove that $\lim_{t\to 0} f(t)$ exists. [Hint: consider least upper bounds.]

4. Let $f\colon [a,b] \to R$ be a continuous function defined on the interval $[a,b]$, where $a < b$.

   (a) Prove that there exists some constant $K > 0$ such that $|f(t)| \leq K$ for all $t \in [a,b]$.

   (b) Prove that there exist real numbers $u$ and $v$ in the interval $[a,b]$ such that $f(u) \leq f(t) \leq f(v)$ for all $t \in [a,b]$.

# Course 121, 1991–92, Test III (JF Hilary Term)

## Attempt THREE questions Answer Question 1 and TWO other questions

1. (a) Let $a$ be a real number, and let $f$ be a real-valued function defined around $a$. What is meant by saying that the function $f$ is *differentiable* at $a$.

   (b) Let $a$ be a real number, and let $f$ and $g$ be real-valued functions defined around $a$. Suppose that $f$ and $g$ are differentiable at $a$. Prove that the product $f.g$ of the functions $f$ and $g$ is differentiable at $a$, and derive the *Product Rule* giving a formula for the derivative $(f.g)'(a)$ of $f.g$ at $a$ in terms of the values of the functions $f$ and $g$ and their derivatives at $a$.

   (c) Evaluate the following limits, using *l'Hôpital's Rule*:

   $$\lim_{t\to 0}\frac{\sin\sin t}{(\sin t)^2}, \qquad \lim_{t\to 1}\frac{t^4-t^3-t+1}{t^3-t^2-t+1}, \qquad \lim_{t\to 0}\frac{\cos(t^2)-1}{t^4}.$$

   (d) State *Taylor's Theorem* (with remainder term).

   (e) Use Taylor's Theorem and the identities

   $$\frac{d}{dt}(\sin t) = \cos t, \qquad \frac{d}{dt}(\cos t) = -\sin t$$

   to prove that

   $$\cos t = \lim_{m\to+\infty}\sum_{n=0}^{m}\frac{(-1)^n t^{2n}}{(2n)!}.$$

2. (a) State and prove *Rolle's Theorem*.

   (b) Let $f\colon \mathbb{R}\to\mathbb{R}$ be a 4 times differentiable function on $\mathbb{R}$. Let $a$, $b$ and $c$ be real numbers satisfying $a < b < c$. Suppose that

   $$f(a) = f(b) = f'(b) = f''(b) = f(c) = 0.$$

   Prove that there exists some real number $s$ satisfying $a < s < c$ for which $f^{(4)}(s) = 0$.

   (c) State the *Mean Value Theorem*, and show how it may be derived from *Rolle's Theorem*.

3. Prove that the polynomial $t^4 + t^2 - 7t - 2$ has exactly 2 distinct real roots, where one of these roots is positive and the other is negative.

4. Prove *Taylor's Theorem*.

# Course 121, 1991–92, Test IV (JF Trinity Term)

## Answer Questions 1 and 2 and ONE other question.

1. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function defined on the interval $[a,b]$.

   (a) Define the concept of a *partition* of the interval $[a,b]$. Give the definition of the *lower sum* $L(P,f)$ and the *upper sum* $U(P,f)$ of $f$ for the partition $P$.

   (b) Define the *lower Riemann integral* $\mathcal{L}\int_a^b f(t)\,dt$ and the *upper Riemann integral* $\mathcal{U}\int_a^b f(t)\,dt$ of $f$ on the interval $[a,b]$. Define precisely what is meant by saying that the function $f$ is *Riemann-integrable* on $[a,b]$, and define the *Riemann integral* of a Riemann-integrable function on $[a,b]$.

   (c) Explain why $L(P,f) \leq U(Q,f)$ for all partitions $P$ and $Q$ of $[a,b]$, and show that
   $$\mathcal{L}\int_a^b f(t)\,dt \leq \mathcal{U}\int_a^b f(t)\,dt.$$
   [You may use, without proof, the fact that $L(R,f) \geq L(P,f)$ and $U(R,f) \leq L(P,f)$ for any refinement $R$ of a partition $P$ of the interval $[a,b]$.]

   (d) Give an example of a bounded function $f\colon [0,1] \to \mathbb{R}$ on the interval $[0,1]$ that is not Riemann-integrable on $[0,1]$.

   (e) Let $f\colon [0,1] \to \mathbb{R}$ be the function on $[0,1]$ defined by
   $$f(t) = \begin{cases} 1 & \text{if } t = \frac{1}{2}; \\ 0 & \text{if } t \neq \frac{1}{2}. \end{cases}$$
   Let $\varepsilon$ satisfy $0 < \varepsilon < 1$. Give an example of a partition of the interval $[0,1]$ into 3 subintervals (not necessarily of equal length) such that $L(P,f) = 0$ and $U(P,f) = \varepsilon$. What are the upper and lower Riemann integrals of the function $f$ on $[a,b]$. Is the function Riemann-integrable on $[a,b]$?

2. (a) What is meant by saying that a infinite sequence $z_1, z_2, z_3, \ldots$ of complex numbers is *convergent*?

   (b) Suppose that $z_n = x_n + iy_n$ for all natural numbers $n$, where $x_n$ and $y_n$ are real numbers and $i = \sqrt{-1}$. Suppose that the sequence $x_1, x_2, x_3, \ldots$ converges to $\lambda$ and that the sequence $y_1, y_2, y_3, \ldots$ converges to $\mu$, where $\lambda$ and $\mu$ are real numbers. Prove that the sequence $z_1, z_2, z_3, \ldots$ converges to the complex number $\lambda + i\mu$.

   (c) What is meant by saying that an infinite series $\sum_{n=1}^{+\infty} s_n$ is convergent?

   (d) For each of the following sequences, state (without proof) whether it is convergent or divergent:—
   $$\sum_{n=1}^{+\infty} \frac{1}{n}, \qquad \sum_{n=1}^{+\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{+\infty} z^n \quad (|z| < 1), \qquad \sum_{n=1}^{+\infty} z^n \quad (|z| \geq 1)$$

(e) Test the following infinite sequences for convergence:—

$$\sum_{n=1}^{+\infty} \frac{5\sin n - 8}{\sqrt{n^3 + 2n^2}}, \qquad \sum_{n=1}^{+\infty} \frac{5\sin n - 8}{\sqrt{n^2 + n}},$$

$$\sum_{n=1}^{+\infty} \frac{z^n(2n+1)}{n!}, \qquad \sum_{n=1}^{+\infty} \frac{(2n)!}{(8n)^n n!}.$$

3. (a) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the interval $[a,b]$, and let

$$F(x) = \int_a^x f(t)\,dt, \qquad (a \le x \le b).$$

Prove that $F'(x) = f(x)$ for all $x$ satisfying $a < x < b$. (N.B., this is part of the *Fundamental Theorem of Calculus.*)

(b) Use the rule for Integration by Parts to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} t^2 \sin t\,dt$.

4. (a) Give the definition of a *Cauchy sequence* of complex numbers.

(b) State the *Bolzano-Weierstrass Theorem* for infinite sequences of complex numbers.

(c) Prove that every Cauchy sequence of complex numbers is bounded.

(d) Prove that every Cauchy sequence of complex numbers is convergent (*Cauchy's Criterion for convergence*). [You may use, without proof, the Bolzano-Weierstrass Theorem.]