Course 121, 1990–91, Test I (JF Michaelmas Term)

Attempt THREE questions

1. (a) Let D be a set of real numbers. Define what is meant by saying that the set D is *bounded above*, and define precisely what is meant by saying that some real number c is the *least upper bound* (or *supremum*) of the set D. State the *Least Upper Bound Axiom*.

   (b) Consider the set D of all real numbers of the form $(n^2 - 1)/(n + 1)^2$ for some natural number n. Is this set bounded above? If so, what is its least upper bound?

   (c) Define what is meant by saying that a real number s belonging to some set D is an *isolated point* of that set. Let D be the set of all real numbers of the form $1/n$ for some natural number n. Prove that every element of D is an isolated point of D.

2. (a) Let $t_1, t_2, t_3, \ldots$ be an infinite sequence of real numbers, and let l be a real number. Define precisely what is meant by saying that the sequence $t_1, t_2, t_3, \ldots$ *converges* to l.

   (b) Let $(s_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ be convergent sequences of real numbers. Prove that the sequence $(s_n - t_n : n \in \mathbb{N})$ is convergent and that
$$\lim_{n\to+\infty}(s_n - t_n) = \lim_{n\to+\infty} s_n - \lim_{n\to+\infty} t_n$$

   (c) Let $s_1, s_2, s_3, \ldots$ be the sequence of real numbers given by
$$s_n = \frac{(n+3)(2n^2-7)}{(3n-1)(5-2n^2)}.$$
Prove that $\lim_{n\to+\infty} s_n = -\frac{1}{3}$. (N.B., *do not attempt to apply l'Hôpital's Rule*)

3. (a) Prove that every convergent sequence of real numbers is bounded.

   (b) Let $t_1, t_2, t_3, \ldots$ be a non-decreasing sequence of real numbers (so that $t_j \leq t_k$ whenever $j < k$). Suppose that this sequence is bounded above. Using the Least Upper Bound Axiom, prove that this sequence converges to some real number l.

   (c) Give an example of a bounded sequence of real numbers which is not convergent.

4. (a) Let $s_1, s_2, s_3, \ldots$ be the sequence of real numbers defined by
$$s_n = \frac{\sqrt{7n-3}\cos(n^3)}{4n+5}$$
Prove that the sequence $(s_n : n \in \mathbb{N})$ is convergent, and find $\lim_{n\to+\infty} s_n$.

   (b) Let $t_1, t_2, t_3, \ldots$ be a sequence of real numbers. Suppose that $t_{n+1}/t_n \to 0$ as $n \to +\infty$. Let ρ be a real number satisfying $0 < \rho < 1$. Prove that there exists a positive constant C such that $|t_n| \leq C\rho^n$ for all natural numbers n. Hence prove that $t_n \to 0$ as $n \to +\infty$. [You may make use of any result proved in lectures, provided that the result is clearly stated.]

Course 121, 1990–91, Test II (JF Michaelmas Term)

Attempt THREE questions

1. (a) Let $f: D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s be a limit point of D. Let l be a real number. Define precisely what is meant by saying that l is the *limit* of $f(t)$ as t tends to s in D.

   (b) Using the definition of the limit of a real-valued function, prove formally that
$$\lim_{t \to 0} \left(t^2 \sin\left(\frac{1}{t^2}\right)\right) = 0.$$

2. (a) Define *precisely* what it means to say that a real-valued function $f: D \to \mathbb{R}$ defined over some subset D of $\mathbb{R}$ is *continuous* at some real number s belonging to D.

   (b) Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be real-valued functions defined over some subset D of $\mathbb{R}$. Suppose that the functions f and g are continuous at some real number s belonging to D. Prove that the sum $f + g$ of the functions f and g is also continuous at s.

   (c) Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be real-valued functions defined over some subset D of $\mathbb{R}$. Suppose that the functions f and g are continuous at some real number s belonging to D. Prove that the product $f.g$ of the functions f and g is also continuous at s (where $(f.g)(t) = f(t)g(t)$ for all $t \in D$).

3. Let $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$ and $h: D \to \mathbb{R}$ be real-valued functions defined over some subset D of $\mathbb{R}$. Let s be some real number belonging to D. Suppose that $f(t) \leq g(t) \leq h(t)$ for all $t \in D$ and that $f(s) = g(s) = h(s)$. Suppose also that the functions f and h are continuous at s. Prove that the function g is continuous at s.

4. State and prove the *Intermediate Value Theorem*.

Course 121, 1990–91, Test III (JF Michaelmas Term)

Attempt THREE questions

1. (a) State and prove *Rolle's Theorem*.

   (b) Let $f: \mathbb{R} \to \mathbb{R}$ be a 5 times differentiable function on $\mathbb{R}$. Let a, b, c and d be real numbers satisfying $a < b < c < d$. Suppose that

$$f(a) = f(b) = f'(b) = f(c) = f'(c) = f(d) = 0.$$

   Prove that there exists some real number s satisfying $a < s < d$ for which $f^{(5)}(s) = 0$.

2. (a) State the *Mean Value Theorem*, and show how it may be derived from *Rolle's Theorem*.

   (b) Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose that $f(0) = a$, $f'(0) = b$ and $f''(t) \geq -c$ for all $t > 0$, where $c > 0$. Prove that $f(t) > a + bt - ct^2$ for all $t > 0$.

   (c) Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $g'(t) \geq 0$ for all $t \in [a, b]$, where a and b are real numbers satisfying $a < b$. Suppose also that the derivative g' of g is continuous and that $g'(t) > 0$ for at least one value of t in the interval (a, b). Prove that $g(b) > g(a)$.

3. Let $f: [a, b] \to \mathbb{R}$ be a bounded function defined on the interval $[a, b]$.

   (a) Define the concept of a *partition* of the interval $[a, b]$. Give the definition of the *lower sum* $L(P, f)$ and the *upper sum* $U(P, f)$ of f for the partition P.

   (b) Define the *lower Riemann integral* $\mathcal{L} \int_a^b f(t)\,dt$ and the *upper Riemann integral* $\mathcal{U} \int_a^b f(t)\,dt$ of f on the interval $[a, b]$. Define precisely what is meant by saying that the function f is *Riemann-integrable* on $[a, b]$, and define the *Riemann integral* of a Riemann-integrable function on $[a, b]$.

   (c) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = e^{kt}$, where $k > 0$, and let a and b be real numbers satisfying $a < b$. Calculate $L(P_n, f)$ and $U(P_n, f)$, where P_n denotes the partition of $[a, b]$ into n subintervals of equal length (so that $P = \{t_0, t_1, \ldots, t_n\}$, where $t_i = a(n - i)/n + bi/n$ for $i = 0, 1, \ldots, n$). Show that

$$\lim_{n \to +\infty} L(P_n, f) = \lim_{n \to +\infty} L(P_n, f) = \frac{1}{k}(e^{kb} - e^{ka}),$$

   and hence show that

$$\mathcal{L} \int_a^b f(t)\,dt = \mathcal{U} \int_a^b f(t)\,dt = \frac{1}{k}(e^{kb} - e^{ka}).$$

   [You may use, without proof, the following identities:

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1} \qquad (x \neq 1),$$

$$\lim_{n \to +\infty} n(e^{c/n} - 1) = c.]$$

4. (a) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the interval $[a,b]$, and let

$$F(x) = \int_a^x f(t)\,dt, \qquad (a \leq x \leq b).$$

Prove that $F'(x) = f(x)$ for all x satisfying $a < x < b$.

(b) Find the derivative of the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \sin\left(\int_0^{1+x^2} e^{t^2}\,dt\right).$$

Course 121, 1990–91, Test IV (JF Trinity Term)

Friday 5th April 1991
Attempt question 1 and 2 other questions

1. Test the following infinite series for convergence:—

$$(a)\quad \sum_{n=1}^{+\infty} \frac{\sin n^3 - 4}{n^3+n}, \qquad (b)\quad \sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt{n\log n}}, \qquad (c)\quad \sum_{n=2}^{+\infty} \frac{1}{\sqrt{n\log n}},$$

$$(d)\quad \sum_{n=1}^{+\infty} \frac{n!z^n}{(2n)!}, \qquad (z\in\mathbb{C}), \qquad (e)\quad \sum_{n=1}^{+\infty} \frac{n!2^n}{(3n)^n}.$$

2. (a) Give the definition of a *Cauchy sequence* of complex numbers.

   (b) Prove that every Cauchy sequence of complex numbers is convergent (*Cauchy's Criterion for convergence*). [You may use, without proof, the Bolzano-Weierstrass Theorem.]

   (c) Describe the *Comparison Test*, used for testing infinite series of complex numbers for convergence, and explain how it can be derived using Cauchy's Criterion for Convergence.

3. (a) Define what is meant by saying that a sequence $z_1, z_2, z_3, z_4, \ldots$ of complex numbers is *convergent*.

   (b) Let $f\colon D \to \mathbb{C}$ be a complex-valued function defined over a subset D of $\mathbb{C}$. What is meant by saying that the function f is *continuous* at an element w of D?

   (c) Let $f\colon D \to \mathbb{C}$ be a complex-valued function defined over a subset D of $\mathbb{C}$, and let $z_1, z_2, z_3, \ldots$ be a sequence of elements of D which converges to some element w of D. Suppose that the function f is continuous at w. Prove that the sequence $f(z_1), f(z_2), f(z_3), \ldots$ converges to $f(w)$.

   (d) Let $f\colon D \to \mathbb{C}$ be a complex-valued function defined over a subset D of $\mathbb{C}$, and let w be a limit point of D. Let l be a complex number. What is meant by saying that l is the *limit* $\lim_{z\to w} f(z)$ of $f(z)$ as $z \to w$ in D?

   (e) Let $f\colon D \to \mathbb{C}$ be a complex-valued function defined over a subset D of $\mathbb{C}$, and let $z_1, z_2, z_3, \ldots$ be a sequence of elements of D which converges to some limit point w of D. Let l be a complex number. Suppose that $\lim_{z\to w} f(z) = l$ for some complex number l and that $z_n = w$ for at most finitely many values of n. Prove that the sequence $f(z_1), f(z_2), f(z_3), \ldots$ converges to l.

   (f) Give an example of a function $f\colon \mathbb{C} \to \mathbb{C}$ and a sequence $z_1, z_2, z_3, \ldots$ of complex numbers converging to 0 such that $\lim_{z\to w} f(z) = 0$, yet the sequence $f(z_1), f(z_2), f(z_3), \ldots$ does not converge to 0.

4. (a) Let $f_1, f_2, f_3, \ldots$ be a sequence of complex-valued functions on a subset D of $\mathbb{C}$. What is meant by saying that the sequence $f_1, f_2, f_3, \ldots$

converges *uniformly* on D to a function f. Explain why the condition that $\lim_{n\to+\infty} f_n(z) = f(z)$ for all $z \in D$ is not in itself sufficient to ensure the uniform convergence of the sequence $f_1, f_2, f_3, \ldots$ to the function f.

(b) Describe the *Weierstrass M-test* for uniform convergence.

(c) Let R_0 be the radius of convergence of the power series $\sum_{n=0}^{+\infty} a_n z^n$. (Thus R_0 is the supremeum, or least upper bound, of the set of all values of $|z|$ corresponding to complex numbers z for which $\sum_{n=0}^{+\infty} a_n z^n$ converges, provided that this set is bounded; otherwise $R_0 = +\infty$.) Let R satisfy $0 < R < R_0$. Prove that the power series $\sum_{n=0}^{+\infty} a_n z^n$ converges uniformly on the open disk $\{z \in \mathbb{C} : |z| < R\}$ of radius R about zero.

Course 121, 1990–91, Test V (JF Trinity Term)

Friday 3rd May 1991.
Attempt question 1 and 2 other questions.
Question 1 carries 30 marks (50%); the remaining questions carry 15 marks (25%) each.

1. Consider the following subsets of $\mathbb{R}^2$. Determine which are open and which are closed in $\mathbb{R}^2$. [Fully justify your answers.]

   (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 25\}$,
   (b) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 25 \text{ or } y > 3\}$,
   (c) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 25 \text{ and } y > 3\}$,
   (d) $\{(x, y) \in \mathbb{R}^2 : 1 < x < 2 \text{ and } 3 < y < 4\}$,
   (e) $\{(x, y) \in \mathbb{R}^2 : y > 0 \text{ and } y^4 = 1 + x^4\}$.

2. (a) Define what is meant by saying that a function $f: \mathbb{R}^n \to \mathbb{R}^m$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is *continuous*. Define also what is meant by saying that a subset V of $\mathbb{R}^n$ is *open*.

   (b) Prove that a function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f^{-1}(V)$ is open in $\mathbb{R}^n$ for every open set V in $\mathbb{R}^m$. (Here $f^{-1}(V)$ denotes the preimage of the set V, defined by
   $$f^{-1}(V) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in V\}.)$$

   (c) Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a continuous function from $\mathbb{R}^3$ to $\mathbb{R}$. Explain why
   $$\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) < z\}$$
   is an open set in $\mathbb{R}^3$.

3. (a) Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Explain what is meant by saying that the sets X and Y are *homeomorphic*.

   (b) Let C be the subset of $\mathbb{R}^2$ defined by
   $$C = \{(x, y) \in \mathbb{R}^2 : x \geq 1 \text{ and } x^2 - y^2 = 1\}.$$
   Prove that C is homeomorphic to the real line $\mathbb{R}$.

   (c) Let S be the subset of $\mathbb{R}^3$ defined by
   $$S = \{(x, y, z) \in \mathbb{R}^3 : x \geq 1 \text{ and } x^2 - y^2 - z^2 = 1\}.$$
   Prove that S is homeomorphic to the Euclidean plane $\mathbb{R}^2$.

4. Let X be a subset of $\mathbb{R}^n$, let $\mathbf{x}$ be a point of X, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that does not belong to X. Let S be the set of all non-negative real numbers t with the property that
   $$(1 - \theta)\mathbf{x} + \theta\mathbf{p} \in X \text{ for all } \theta \in [0, t]$$
   (i.e., S is the set of all non-negative real numbers t with the property that the line segment joining the point $\mathbf{x}$ to the point $(1-t)\mathbf{x} + t\mathbf{p}$ is contained within the set X). Let $s = \sup S$, and let $\mathbf{y} = (1 - s)\mathbf{x} + s\mathbf{p}$.

(a) Explain why $0 \leq s \leq 1$.

(b) Show that if the set X is closed in $\mathbb{R}^n$ then the point $\mathbf{y}$ belongs to X.

(c) Show that if the set X is open in $\mathbb{R}^n$ then the point $\mathbf{y}$ belongs to $\mathbb{R}^n \setminus X$.

(d) Using (a) and (b), show that the only subsets of $\mathbb{R}^n$ that are both open and closed in $\mathbb{R}^n$ are the empty set $\emptyset$ and $\mathbb{R}^n$ itself.