Course 121, 1990–91, Supplemental Examination (JF)

1. (a) What is meant by saying that a subset D of the set $\mathbb{R}$ of real numbers is *bounded above*? What is meant by saying that a real number s is the *least upper bound* (or *supremum*) $\sup D$ of the set D?

   (b) State the *Least Upper Bound Axiom* satisfied by the real number system.

   (c) For each of the following sets, determine whether or not the set is bounded above and, if so, determine the least upper bound of the set [giving appropriate justification for your answers]:

   (i) the set $\mathbb{Z}$ of all integers,

   (ii) the set $\left\{\dfrac{n-2}{n} : n \in \mathbb{N}\right\}$ consisting of all real numbers that are of the form $(n-2)/n$ for some natural number n,

   (iii) the set of all rational numbers q satisfying $q^2 < 9$.

2. (a) Define precisely what is meant by saying that an infinite sequence $t_1, t_2, t_3, \ldots$ of real numbers *converges* to some real number l.

   (b) Let (s_n) and (t_n) be convergent sequences of real numbers. Prove that the sequence $(s_n + t_n)$ is convergent, and

$$\lim_{n\to+\infty}(s_n + t_n) = \lim_{n\to+\infty} s_n + \lim_{n\to+\infty} t_n.$$

   (c) Let $(s_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ be the infinite sequences of real numbers defined by

$$s_n = \frac{9n^2 - 7}{3n^2 + 2n}, \qquad t_n = \frac{7}{n^3}\cos n.$$

   Prove that the infinite sequences (s_n) and (t_n) are convergent, and find the limits of these sequences.

3. (a) Define precisely what is meant by saying that a function $f: D \to \mathbb{R}$ is *continuous* at s, where $s \in D$.

   (b) Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) = \begin{cases} \sqrt{t}\sin(1/t) & \text{if } t > 0; \\ 0 & \text{if } t \leq 0. \end{cases}$$

Using the formal definition of continuity, prove that the function f is continuous at 0.

(c) Let f, g and h be real-valued functions defined on some subset D of the set $\mathbb{R}$ of real numbers. Suppose that $f(t) \leq g(t) \leq h(t)$ for all $t \in D$, and that $f(s) = g(s) = h(s)$ for some $s \in D$. Suppose also that the functions f and h are continuous at s. Using the formal definition of continuity, prove that the function g is continuous at s.

4. State and prove the *Intermediate Value Theorem.*

5. (a) State and prove *Rolle's Theorem.*

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be a 5-times differentiable function. Let a, b and c be real numbers satisfying $a < b < c$. Suppose that

$$f(a) = f'(a) = f(b) = f'(b) = f(c) = f'(c) = 0.$$

Prove that there exists some s satisfying $a < s < c$ for which $f^{(5)}(s) = 0$.

(c) State the *Mean Value Theorem*, and prove it, using Rolle's Theorem.

6. (a) Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. Define the *upper sum* $U(P, f)$ and the *lower sum* $L(P, f)$ of f for a partition P of $[a, b]$. Define the upper and lower Riemann integrals of f on $[a, b]$. What is meant by saying that the function f is *Riemann-integrable* on $[a, b]$? What is the *Riemann integral* of a Riemann-integrable function f on $[a, b]$?

(b) Suppose that the function f is constant, with value c. What are the values of the upper sum $U(P, f)$ and the lower sum $L(P, f)$ of f for any partition P of the interval $[a, b]$? What are the values of the upper and lower Riemann integrals of f on $[a, b]$?

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 1 & \textit{if } t \textit{ is rational;} \\ 0 & \textit{if } t \textit{ is irrational.} \end{cases}$$

Explain why the function f is not Riemann-integrable on the interval $[0, 1]$.

7. Test the following infinite series for convergence:

$$(i)\quad \sum_{n=1}^{+\infty} \frac{3\sin n - 2}{n\sqrt{n}}, \qquad (ii)\quad \sum_{n=1}^{+\infty} \frac{5 + 3\cos n}{n},$$

$$(iii)\quad \sum_{n=1}^{+\infty} \frac{1}{(3n)!}, \qquad (iv)\quad \sum_{n=1}^{+\infty} \frac{n!}{n^{n+2}}.$$

8. Let $f_1, f_2, f_3, \ldots$ be a sequence of complex-valued functions defined over some subset D of the complex plane $\mathbb{C}$.

   (a) What is meant by saying that the sequence $f_1, f_2, f_3, \ldots$ of functions converges *uniformly* on D to some function $f: D \to \mathbb{C}$?

   (b) Suppose that each function f_n is continuous and that the sequence $f_1, f_2, f_3, \ldots$ converges uniformly on D to the function f. Prove that the limit function f is continuous.

9. (a) Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. What is meant by saying that a function $\varphi: X \to Y$ is *continuous* at some point $\mathbf{p}$ of X?

   (b) What is meant by saying that a subset U of X is *open* in X?

   (c) Prove that the function $\varphi: X \to Y$ is continuous if and only if, given any open set V in Y, the preimage $\varphi^{-1}(V)$ of V is open in X.

10. Consider the following subsets of $\mathbb{R}^3$. Determine which are open and which are closed in $\mathbb{R}^3$. [Fully justify your answers.]

   (i) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 5\}$,

   (ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 2 \text{ and } z > 1\}$,

   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 2 \text{ or } z \leq 1\}$,

   (iv) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 5 \text{ or } z < 1\}$.