Course 121, 1989–90, Test III (JF Hilary Term)

Friday 2nd February 1990, 3.00–4.30pm
Answer any THREE questions

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathbb{R}$. State the *Product Rule* and the *Quotient Rule* for differentiating $f.g$ and f/g respectively, and prove these rules, using standard properties of limits.

2. (a) State *Rolle's Theorem.*

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 5-times differentiable function. Let a, b and c be real numbers satisfying $a < b < c$. Suppose that

   $$f(a) = f'(a) = f(b) = f'(b) = f(c) = f'(c) = 0.$$

   Prove that there exists some s satisfying $a < s < c$ for which $f^{(5)}(s) = 0$.

   (c) State the *Mean Value Theorem*, and show how it may be derived from Rolle's Theorem.

3. (a) State *Cauchy's Mean Value Theorem*, and show how *l'Hôpital's Rule* may be derived from it.

   (b) Evaluate the following limits, using l'Hôpital's Rule:

   $$\lim_{t\to 0} \frac{\sin\sin\sin t}{t}, \qquad \lim_{t\to 5} \frac{t^3 - 12t^2 + 45t - 50}{t^3 - 9t^2 + 15t + 25}, \qquad \lim_{t\to 0} \frac{\cos(t^2) - 1}{\sin t^4}.$$

   (a) State and prove Taylor's Theorem.

Course 121, 1989–90, Test IV (JF Hilary Term)

Friday 2nd March 1990, 3.00–4.30pm
Attempt question 1 and 2 other questions

1. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function defined on the interval $[a,b]$.

   (a) Define the concept of a *partition* of the interval $[a,b]$. Give the definition of the *lower sum* $L(P,f)$ and the *upper sum* $U(P,f)$ of f for the partition P.

   (b) Define the *lower Riemann integral* $\mathcal{L}\int_a^b f(t)\,dt$ and the *upper Riemann integral* $\mathcal{U}\int_a^b f(t)\,dt$ of f on the interval $[a,b]$. Define precisely what is meant by saying that the function f is *Riemann-integrable* on $[a,b]$, and define the *Riemann integral* of a Riemann-integrable function on $[a,b]$.

   (c) Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(t) = 1 - t^2$. Calculate $L(P_n,f)$ and $U(P_n,f)$, where P_n denotes the partition of $[0,1]$ into n subintervals of length $1/n$ (i.e., $P = \{t_0, t_1, \ldots, t_n\}$, where $t_i = i/n$ for $i = 0, 1, \ldots, n$). Hence show that

   $$\mathcal{L}\int_0^1 f(t)\,dt = \mathcal{U}\int_0^1 f(t)\,dt = \frac{2}{3}.$$

   [You may use, without proof, the following identities:

   $$\sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1), \qquad \sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1).]$$

2. (a) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the interval $[a,b]$, and let

   $$F(x) = \int_a^x f(t)\,dt, \qquad (a \le x \le b).$$

   Prove that $F'(x) = f(x)$ for all x satisfying $a < x < b$.

   (b) Find the derivative of the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

   $$g(x) = \int_0^{x^4} t^2 e^{t^2}\,dt.$$

3. (a) Let f be a function that is k times differentiable and whose kth derivative is continuous on some open interval containing the real numbers a and $a+h$. Using the rule for integration by parts, show that

   $$f(a+h) = f(a) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(a) + r_k(a,h),$$

   where

   $$r_k(a,h) = \frac{h^k}{(k-1)!}\int_0^1 (1-t)^{k-1} f^{(k)}(a+th)\,dt.$$

(b) Show that

$$\log(1+h) = \lim_{k\to+\infty} \sum_{n=1}^{k} \frac{(-1)^{(n-1)} h^n}{n}$$

for all h satisfying $-1 < h < 1$.

4. Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions on an interval $[a, b]$, and let f be a real-valued function on $[a, b]$.

   (a) Define what is meant by saying that the functions f_n converge *uniformly* to f on the interval $[a, b]$ as $n \to +\infty$.

   (b) Suppose that the functions f_n converge uniformly to f on $[a, b]$ as $n \to +\infty$. Prove that

   $$\lim_{n\to+\infty} \int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt.$$

   (c) Give an example of a sequence $f_1, f_2, f_3, \ldots$ of continuous real-valued functions on an interval $[a, b]$ and a continuous real-valued function f on $[a, b]$ such that

   $$\lim_{n\to+\infty} \int_a^b f_n(t)\,dt \neq \int_a^b f(t)\,dt,$$

   even though $\lim_{n\to+\infty} f_n(t) = f(t)$ for all $t \in [a, b]$.

Course 121, 1989–90, Test V (JF Trinity Term)

Friday 20th April 1990, 3.00–4.30pm
Attempt question 1 and 2 other questions

1. Test the following infinite series for convergence:—

$$(a) \quad \sum_{n=1}^{+\infty} \frac{z^n}{n!\sqrt{n}} \quad (z \in \mathbb{C}), \qquad (b) \quad \sum_{n=1}^{+\infty} \frac{3\sin n - 2}{n\sqrt{n}}, \qquad (c) \quad \sum_{n=2}^{+\infty} \frac{(-1)^n}{\log n},$$

$$(d) \quad \sum_{n=1}^{+\infty} \frac{4 + \cos n^2}{n}, \qquad (e) \quad \sum_{n=1}^{+\infty} \frac{n!}{n^{n+2}}.$$

2. (a) Define what is meant by saying that a sequence $z_1, z_2, z_3, z_4, \ldots$ of complex numbers is a *Cauchy sequence*.

   (b) Prove that every convergent sequence of complex numbers is a Cauchy sequence.

   (c) Prove the every Cauchy sequence of complex numbers is bounded.

   (d) Prove that every Cauchy sequence of complex numbers is convergent. [You may use, without proof, the Bolzano-Weierstrass Theorem, which states that every bounded sequence of complex numbers has a convergent subsequence.]

3. (a) Prove that the infinite series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent

   (b) By using the same method as in (a), or otherwise, prove that the infinite series $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n \log n}$ is divergent

4. (a) State the *Alternating Series Test*, and prove that any infinite series satisfying the conditions of this test is convergent.

   (b) Does the infinite series

$$\sum_{n=1}^{+\infty} \frac{2\cos n\pi + \sin \frac{1}{2}n\pi}{n^2}$$

   satisfy the conditions of the Alternating Series Test? Is this infinite series convergent?

Course 121, 1989–90, Test VI (JF Trinity Term)

Friday 11th May 1990, 3.00–4.30pm
Attempt question 1 and 2 other questions

1. Determine which of the following subsets of the complex plane are open and which are closed:—

   (a) $\{z \in \mathbb{C} : |z+2| < 7\}$,
   (b) $\{z \in \mathbb{C} : |z+2| > 7\}$,
   (c) $\{z \in \mathbb{C} : |z+2| \geq 7 \text{ and } \operatorname{Re} z \leq 0\}$,
   (d) $\{z \in \mathbb{C} : |z+2| \geq 7 \text{ and } \operatorname{Re} z < 0\}$,
   (e) $\{z \in \mathbb{C} : |\exp z + z^3| < 7\}$.
   [Briefly justify your answers.]

2. (a) Prove that a sequence $z_1, z_2, z_3, \ldots$ of complex numbers converges to some complex number l if and only if, given any open set U which contains l, there exists some natural number N such that the point z_j belongs to U for all j satisfying $j \geq N$.

   (b) Using (a), or otherwise, show that if F is a closed set in the complex plane, and if $z_1, z_2, z_3, z_4, \ldots$ is an infinite sequence of complex numbers belonging to F which converges to some complex number l then $l \in F$.

3. (a) Let K be a closed bounded subset of the complex plane and let $f\colon K \to \mathbb{C}$ be a continuous function on K. Prove that there exists some non-negative real number C such that $|f(z)| \leq C$ for all $z \in K$.

   (b) Let K be a closed bounded subset of the complex plane and let $f\colon K \to \mathbb{C}$ be a continuous function on K. Let w be a complex number with the property that $f(z) \neq w$ for all $z \in K$. Prove that there exists some real number $\delta > 0$ such that $|f(z) - w| \geq \delta$ for all $z \in K$.

4. Let $\sum_{n=0}^{+\infty} a_n z^n$ be a power series whose coefficients $a_0, a_1, a_2, \ldots$ are complex numbers.

   (a) Define the *radius of convergence* of this power series.

   (b) Prove that the power series $\sum_{n=0}^{+\infty} a_n z^n$ converges if $z < R_0$, but diverges if $z > R_0$, where R_0 is the radius of convergence of the power series.