Course 121, 1988–89, Test I (JF Michaelmas Term)

1. Let S be a set of real numbers.

   (a) Define what is meant by saying that the set S is *bounded above*, and define precisely what is meant by saying that some real number c is the *least upper bound* (or *supremum*) of the set S. State the *Least Upper Bound Axiom*.

   Let S be a set of real numbers, and let c be some real number (which may or may not belong to S).

   > The real number c is said to be a *cluster point* of the set S if, for every $\delta > 0$, there exists some real number s belonging to S which satisfies $|s - c| < \delta$.
   >
   > The real number c is said to be a *limit point* of the set S if, for every $\delta > 0$, there exists some real number s belonging to S which satisfies both $|s - c| < \delta$ and $s \neq c$.
   >
   > The real number c is said to be an *isolated point* of the set S if c belongs to S and moreover there exists some positive real number δ_0 with the property that if $s \in S$ satisfies $|s - c| < \delta_0$ then $s = c$.

   Note that every limit point of the set S is a cluster point of S, as is every isolated point of S. Every real number belonging to S is a cluster point of S but it need not necessarily be a limit point of S. Indeed a real number belonging to the set S cannot be both a limit point of S and an isolated point of S.

   (b) Explain why every cluster point c of the set S is either a limit point of S or else is an isolated point of S.

   (c) Prove that if the set S of real numbers is bounded above then the least upper bound $\sup S$ of the set S is a cluster point of the set S.

   (d) Let c be a limit point of some set S of real numbers. Let δ be a positive real number. Prove that the number of elements s of the set S which satisfy $|s - c| < \delta$ is infinite.

2. Let $(s_j : j \in \mathbb{N})$, $(t_j : j \in \mathbb{N})$ and $(u_j : j \in \mathbb{N})$ be sequences of real numbers satisfying $s_j \leq t_j \leq u_j$ for all natural numbers j. Let l be a real number. Suppose that $s_j \to l$ and $u_j \to l$ as $j \to +\infty$. Let ε be any real number satisfying $\varepsilon > 0$. Show that there exist natural numbers N_1 and N_2 such that $t_j < l + \varepsilon$ for all j satisfying $j \geq N_1$ and $t_j > l - \varepsilon$ for all j satisfying $j \geq N_2$. Hence or otherwise prove that $t_j \to l$ as $j \to +\infty$.

3. (a) Let $(t_j : j \in \mathbb{N})$ be a non-decreasing sequence of real numbers which is bounded above. Prove that there exists some real number l such that $t_j \to l$ as $j \to +\infty$.

(b) Let $I_1, I_2, I_3, I_4, I_5, \ldots$ be an infinite sequence of intervals in $\mathbb{R}$, where each interval I_j is given by

$$I_j = [a_j, b_j] \equiv \{t \in \mathbb{R} : a_j \leq t \leq b_j\}$$

for some real numbers a_j and b_j satisying $a_j < b_j$. Suppose that $I_{j+1} \subset I_j$ for each natural number j and that $b_j - a_j \to 0$ as $j \to +\infty$. Prove that there exists exactly one real number c with the property that c belongs to I_j for each natural number j.

Course 121, 1988–89, Test II (JF Michaelmas Term)

1. (a) Define *precisely* what it means to say that a sequence $(t_j : j \in \mathbb{N})$ is a *Cauchy Sequence*.

   (b) State the *Cauchy Criterion for Convergence* (also known as the *General Principle of Convergence*).

   (c) Let u be a real number satisfying $u \neq 1$. Show that

   $$u^j + u^{j+1} + \cdots + u^{k-1} = \frac{u^j - u^k}{1 - u},$$

   for all natural numbers j and k satisfying $j < k$.

   (d) Let $(t_j : j \in \mathbb{N})$ be a sequence of real numbers. Let u be a real number satisfying $0 < u < 1$. Suppose that $|t_j - t_{j+1}| < u^j$ for all natural numbers j. Use (c) in order to show that

   $$|t_j - t_k| \leq \frac{u^j}{1 - u}$$

   for all natural numbers j and k satisfying $j < k$. Explain why the sequence $(t_j : j \in \mathbb{N})$ is a Cauchy sequence. Is the sequence convergent? [You may use without proof the fact that if $0 < u < 1$ then $u^j \to 0$ as $j \to +\infty$.]

2. (a) Define *precisely* what it means to say that a real-valued function $f \colon D \to \mathbb{R}$ defined over some subset D of $\mathbb{R}$ is *continuous* at some real number s belonging to D.

   (b) Let $f \colon D \to \mathbb{R}$ and $g \colon D \to \mathbb{R}$ be real-valued functions defined over some subset D of $\mathbb{R}$. Suppose that the functions f and g are continuous at some real number s belonging to D. Prove that the sum $f + g$ of the functions f and g is also continuous at s.

   (c) Let $f \colon D \to \mathbb{R}$, $g \colon D \to \mathbb{R}$ and $h \colon D \to \mathbb{R}$ be real-valued functions defined over some subset D of $\mathbb{R}$. Let s be some real number belonging to D. Suppose that $f(t) \leq g(t) \leq h(t)$ for all $t \in D$ and that $f(s) = g(s) = h(s)$. Suppose also that the functions f and h are continuous at s. Show that, given any $\varepsilon > 0$, there exist strictly positive real numbers δ_1 and δ_2 such that $g(t) < g(s) + \varepsilon$ whenever $t \in D$ satisfies $|t - s| < \delta_1$ and $g(t) > g(s) - \varepsilon$ whenever $t \in D$ satisfies $|t - s| < \delta_2$. Hence or otherwise, show that the function g is continuous at s.

3. (a) State the *Bolzano-Weierstrass Theorem.*

   (b) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on the interval $[a,b]$, where a and b are real numbers satisfying $a < b$, and where

$$[a,b] \equiv \{t \in \mathbb{R} : a \leq t \leq b\}.$$

   By making use of the *Bolzano-Weierstrass Theorem*, or by making use of the *Least Upper Bound Axiom*, or otherwise, prove that there exists some positive real number K such that $|f(t)| \leq K$ for all $t \in [a,b]$.

Course 121, 1988–89, Test III (JF Hilary Term)

1. For each of the following statements, decide whether the statement is true or false. You should merely answer **TRUE**, **FALSE**, or **DON'T KNOW**. You are not required to justify your answer. *(Note: you may be penalized for an incorrect **TRUE** or **FALSE** answer; an answer of **DON'T KNOW**, or no answer at all, will neither gain nor lose you marks.)*

   (i) if $a_1, a_2, a_3, a_4, \ldots$ is a sequence of real numbers and if $a_n \to 0$ as $n \to +\infty$ then the infinite series $a_1 + a_2 + a_3 + a_4 + \cdots$ is convergent,

   (ii) if $a_1, a_2, a_3, a_4, \ldots$ is a sequence of real numbers and if the infinite series $a_1 + a_2 + a_3 + a_4 + \cdots$ is convergent then $a_n \to 0$ as $n \to +\infty$,

   (iii) the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is convergent,

   (iv) the infinite series $1 + x + x^2 + x^3 + x^4 + \cdots$ is convergent for all values of the real number x,

   (v) the infinite series $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ is convergent for all values of the real number x,

   (vi) if $a_1, a_2, a_3, a_4, \ldots$ is a sequence of real numbers and if $r = \displaystyle\lim_{n\to+\infty} \frac{a_{n+1}}{a_n}$ exists and satisfies $|r| < 1$ then the infinite series $\displaystyle\sum_{n=1}^{+\infty} a_n$ is convergent,

   (vii) if $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ are sequences of real numbers, if $0 \leq a_n \leq b_n$ for all natural numbers n, and if $\displaystyle\sum_{n=1}^{+\infty} a_n$ is divergent then $\displaystyle\sum_{n=1}^{+\infty} b_n$ is also divergent,

   (viii) if $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ are sequences of real numbers, if $a_n \leq b_n$ for all natural numbers n, and if $\displaystyle\sum_{n=1}^{+\infty} b_n$ is convergent then $\displaystyle\sum_{n=1}^{+\infty} a_n$ is also convergent,

(ix) every convergent sequence of real numbers is absolutely convergent,

(x) every absolutely convergent sequence of real numbers is convergent.

2. Test the following infinite series for convergence:—

$$(i)\ \sum_{n=1}^{+\infty} \frac{3\cos n - 2}{2n\sqrt{n} - n}, \qquad (ii)\ \sum_{n=1}^{+\infty} \frac{2\sin n + 5}{n},$$

$$(iii)\ \sum_{n=1}^{+\infty} \frac{\sqrt{n}x^2}{n!}, \qquad (x \in \mathbb{R}), \qquad (iv)\ \sum_{n=1}^{+\infty} \frac{n!}{(2n)^n}.$$

3. Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be real-valued functions defined on the set $\mathbb{R} \setminus \{0\}$ of all non-zero real numbers.

   (a) State *precisely* what is meant by saying that $\lim_{t \to 0} f(t) = l$, where l is some (finite) real number.

   We write $\lim_{t \to 0} g(t) = +\infty$ if, for all real numbers K (no matter how large) there exists some $\delta > 0$ such that $g(t) > K$ for all real numbers t satisfying $0 < |t| < \delta$.

   (b) Prove directly from the relevant definitions that if $\lim_{t \to 0} g(t) = +\infty$ then $\lim_{t \to 0} 1/g(t) = 0$.

   (c) Prove that if $f(t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$, if $\lim_{t \to 0} f(t) = l$ for some non-negative real number l, and if $\lim_{t \to 0} g(t) = +\infty$ then $\lim_{t \to 0} g(t)/f(t) = +\infty$.

Course 121, 1988–89, Test IV (JF Trinity Term)

1. (a) State and prove *Rolle's Theorem*. [You may assume without proof any standard property of continuous functions that you require, provided that any such result is clearly stated.]

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function from $\mathbb{R}$ to $\mathbb{R}$ which is $2k + 1$ times differentiable, for some non-negative integer k. Let a and b be real numbers satisfying $a < b$. Suppose that $f^{(j)}(a) = 0$ and $f^{(j)}(b) = 0$ for $j = 0, 1, \ldots, k$. Prove that there exists some $\xi \in \mathbb{R}$ satisfying $a < \xi < b$ for which $f^{(2k+1)}(\xi) = 0$.

   (c) State the *Mean Value Theorem*, and show how it may be deduced as a corollary of *Rolle's Theorem*.

2. Throughout this question we denote by $B(z, r)$ the *open disk* of radius r about z, where z is any complex number, r is any positive real number, and $B(z, r)$ is the subset of the complex plane defined by

$$B(z, r) = \{w \in \mathbb{C} : |w - z| < r\}.$$

   (a) One of the following statements is a correct definition of the concept of an *open set*, and the others are incorrect. State which of the following is the correct definition:—

I. a subset D of the complex plane is *open* if and only if there exists some element z of D such that $B(z, \delta) \subset D$ for all $\delta > 0$,

II. a subset D of the complex plane is *open* if and only if there exists some element z of D and some $\delta > 0$ such that $B(z, \delta) \subset D$,

III. a subset D of the complex plane is *open* if and only if, given any element z of D, there exists some $\delta > 0$ such that $B(z, \delta) \subset D$,

IV. a subset D of the complex plane is *open* if and only if there exists some $\delta > 0$ such that $B(z, \delta) \subset D$ for all elements z of D.

(b) Let $D_1, D_2, D_3, \ldots, D_k$ be open sets in the complex plane. Prove that their intersection $D_1 \cap D_2 \cap \cdots \cap D_k$ is also open.

(c) Let z_0 be any complex number, and let r be any positive real number. Prove that the open disk $B(z_0, r)$ is an open set in the complex plane. [Hint: use the *Triangle Inequality*, which states that $|z_3 - z_1| \leq |z_3 - z_2| + |z_2 - z_1|$ for all complex numbers z_1, z_2 and z_3.]

(d) Is the set $\{z \in \mathbb{C} : |z + 1| < 2 \text{ and } |z - 1| < 2\}$ an open set? [Justify your answer.]

Recall that if D is a subset of the complex plane then a complex number z belonging to D is said to be an *interior point* of D if and only if there exists some $\delta > 0$ such that $B(z, \delta) \subset D$, and a complex number z (which may or may not belong to D) is said to be a *cluster point* of D if and only if $B(z, \delta) \cap D$ is non-empty for all $\delta > 0$. Let $\operatorname{int} D$ denote the set of all interior points of D (known as the *interior* of D), and let $\overline{D}$ denote the set of all cluster points of D (known as the *closure* of D).

(e) Let D be a subset of the complex plane. Show that

$$\mathbb{C} \setminus \overline{D} = \operatorname{int}(\mathbb{C} \setminus D),$$

where $\mathbb{C} \setminus \overline{D}$ is the complement of the closure $\overline{D}$ of D and $\operatorname{int}(\mathbb{C} \setminus D)$ is the interior of the complement $\mathbb{C} \setminus D$ of D.