Course 121, 1988–89, Annual Examination (JF)

1. (a) Define the concept of the *least upper bound* $\sup S$ for an appropriate set S of real numbers. State the *Least Upper Bound Axiom* satisfied by the real number system.

   (b) Let S and T be subsets of the set of real numbers. Suppose that both S and T are bounded above and that the following property holds: given any element s of S, and given any real number δ satisfying $\delta > 0$, there exists some element t of T for which $|t-s| < \delta$. Prove that $\sup S \leq \sup T$.

2. (a) Let $(t_j : j \in \mathbb{N})$ be an infinite sequence of real numbers. Let l be a real number. Define precisely what is meant by saying that l is the *limit* of the sequence $(t_j : j \in \mathbb{N})$ as $j \to +\infty$.

   (b) Let $(t_j : j \in \mathbb{N})$ and $(u_j : j \in \mathbb{N})$ be sequences of real numbers, and let l and m be real numbers. Suppose that $\lim_{j \to +\infty} t_j = l$ and that $\lim_{j \to +\infty} u_j = m$. Prove that

$$\lim_{j \to +\infty} (t_j + u_j) = l + m.$$

   (c) State the *Bolzano-Weierstrass Theorem.* [No proof is required.]

   (d) Consider the infinite sequence $(t_j : j \in \mathbb{N})$ given by

$$t_j = \begin{cases} \dfrac{j+1}{j} & \text{if } j \text{ is odd;} \\ 2j & \text{if } j \text{ is even.} \end{cases}$$

   Does this sequence have a convergent subsequence? [Justify your answer.]

3. (a) Let $f: D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$. Let s be an element of D. Define precisely what is meant by saying that the function f is *continuous* at s.

   (b) Let D and E be subsets of $\mathbb{R}$, and let $f: D \to \mathbb{R}$ and $g: E \to \mathbb{R}$ be real-valued functions defined over D and E respectively, where $f(D) \subset E$. Let s be an element of D. Suppose that the function f is continuous at s and that the function g is continuous at $f(s)$. Prove that the composition function $g \circ f$ is continous at s (where $(g \circ f)(t) = g(f(t))$ for all $t \in D$).

(c) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) = \begin{cases} 9t\cos^2\left(\dfrac{1}{t^3}\right) & \text{if } t \neq 0; \\ 0 & \text{if } t = 0. \end{cases}$$

Is the function f continuous at 0? [Justify your answer.]

4. Test the following infinite series for convergence:—

$$\sum_{n=1}^{+\infty} \frac{2\cos n - 1}{2n^2 + \sin n}, \qquad \sum_{n=1}^{+\infty} \frac{n+7}{n!},$$

$$\sum_{n=1}^{+\infty} \frac{n+1}{n\sqrt{n}}, \qquad \sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n^2)^n}.$$

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function, and let α be a real number. We recall that $\lim_{t\to+\infty} f(t) = \alpha$ if and only if, for any $\varepsilon > 0$, there exists some real number L such that $|f(t) - \alpha| < \varepsilon$ whenever $t > L$. We recall also that $\lim_{t\to+\infty} f(t) = +\infty$ if and only if, for any real number K, there exists some real number L such that $f(t) > K$ whenever $t > L$. Using these definitions, prove that if $f(t) > 0$ for all $t \in \mathbb{R}$ and if $\lim_{t\to+\infty} f(t) = +\infty$ then

$$\lim_{t\to+\infty} \frac{f(t)}{f(t)+1} = 1.$$

6. (a) Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over a subset D of $\mathbb{R}$, and let a be an interior point of the set D. Define what is meant by saying that the function f is *differentiable* at a, and express the derivative $f'(a)$ of f at a (if it exists) in terms of an appropriate limit.

(b) State the *Product Rule* for differentiating the product of two differentiable functions, and show how this rule can be derived as a consequence of standard properties of limits of real-valued functions.

(c) State the *Quotient Rule* for differentiating the quotient of two differentiable functions (where the denominator of this quotient is everywhere non-zero), and show how this rule can be derived as a consequence of standard properties of limits of real-valued functions.

7. (a) Define what is meant by saying that a subset of the complex plane is an *open set*. Define also what is meant by saying that a subset of the complex plane is a *closed set*.

   (b) Consider the subsets A_1, A_2, A_3 and A_4 of the complex plane $\mathbb{C}$ specified as follows:—

$$\begin{aligned} A_1 &= \{z \in \mathbb{C} : |z - 1| > 2\}, \\ A_2 &= \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z \leq 1\}, \\ A_3 &= \{z \in \mathbb{C} : |z| \geq 3\}, \\ A_4 &= \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } |z| < 2\}. \end{aligned}$$

   Which of these sets are open sets in $\mathbb{C}$? Which of these sets are closed sets in $\mathbb{C}$? [Briefly justify your answers.]

8. (a) Define the concept of the *radius of convergence* R_0 of a power series $\sum_{n=0}^{+\infty} a_n z^n$, and show that this power series converges if $|z| < R_0$ but diverges if $|z| > R_0$.

   (b) Let D be a subset of $\mathbb{C}$, and let $f_0, f_1, f_2, f_3, \ldots$ be a sequence of functions mapping D into $\mathbb{C}$. Explain what is meant by saying that the infinite series $\sum_{n=0}^{+\infty} f_n(z)$ converges *uniformly* on D. State the *Weierstrass M-test* for uniform convergence of an infinite series of functions.

   (c) Show that if R is a positive real number satisfying $R < R_0$, where R_0 is the radius of convergence of the power series $\sum_{n=0}^{+\infty} a_n z^n$, then the power series converges uniformly on the open disk $\{z \in \mathbb{C} : |z| < R\}$.

   (d) Explain briefly why the power series $\sum_{n=0}^{+\infty} a_n z^n$ converges to a continuous function on the disk $\{z \in \mathbb{C} : |z| < R_0\}$, where R_0 is the radius of convergence of this power series. [You may assume without proof any general results concerning uniform convergence that you need, provided that such results are clearly stated.]