Course 121, 1987–88, Supplemental Examination (JF)

[Questions for Michaelmas and Hilary Terms were contributed by another lecturer. Only Trinity Term questions appear here.]

TT1. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be real-valued functions on $\mathbb{R}^n$, let $\mathbf{u}$ be a point of $\mathbb{R}^n$, and let l be a real number.

(a) Define precisely what is meant by saying that $\lim\limits_{\mathbf{x}\to\mathbf{u}} f(\mathbf{x}) = l$. Also define what is meant by saying that $\lim\limits_{\mathbf{x}\to\mathbf{u}} g(\mathbf{x}) = +\infty$.

(b) Suppose that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that

$$\lim_{\mathbf{x}\to\mathbf{u}} f(\mathbf{x}) = 0 \text{ if and only if } \lim_{\mathbf{x}\to\mathbf{u}} \frac{1}{f(\mathbf{x})} = +\infty.$$

(c) Suppose that $f(\mathbf{x}) > 0$ and $f(\mathbf{x})g(\mathbf{x}) \geq 2$ for all $\mathbf{x} \in \mathbb{R}^n$, and that $\lim\limits_{\mathbf{x}\to\mathbf{u}} f(\mathbf{x}) = 0$. Prove that $\lim\limits_{\mathbf{x}\to\mathbf{u}} g(\mathbf{x}) = +\infty$.

TT2. (a) State the Triangle Inequality (for points in $\mathbb{R}^n$).

(b) Define the concept of an *open set* in n-dimensional Euclidean space $\mathbb{R}^n$.

(c) Let $\mathbf{u}$ and $\mathbf{v}$ be points of $\mathbb{R}^n$, and let $B(\mathbf{u}, r)$ and $B(\mathbf{v}, s)$ denote the open balls of radius r and s about the points $\mathbf{u}$ and $\mathbf{v}$ respectively, where

$$B(\mathbf{u}, r) \equiv \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}-\mathbf{u}| < r\}, \qquad B(\mathbf{v}, s) \equiv \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}-\mathbf{v}| < s\}.$$

Show that if $0 < |\mathbf{u} - \mathbf{v}| < r$ and $0 < s \leq r - |\mathbf{u} - \mathbf{v}|$ then $B(\mathbf{v}, s) \subset B(\mathbf{u}, r)$. Hence prove that $B(\mathbf{u}, r)$ is an open set in $\mathbb{R}^n$.

(d) Prove that the subset U of $\mathbb{R}^3$ defined by

$$U = \{(x, y, z) \in \mathbb{R}^3 : y > 0\}$$

is an open set in $\mathbb{R}^3$.

TT3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function on $\mathbb{R}^2$. Suppose that

(i) $f(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$,

(ii) $f(x, y) = 0$ if and only if $(x, y) = (0, 0)$, (*iii*) $f(\lambda x, \lambda y) = \lambda^4 f(x,y)$ for all $(x,y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$,

(*iv*) the function f is continuous on $\mathbb{R}^2$.

By considering the least upper bound of the values of the function f on the ellipse

$$\{(x,y) \in \mathbb{R}^2 : 3x^2 + 4y^2 = 1\},$$

or otherwise, show that there exists a constant C such that $C < 0$ and $f(x,y) \leq C(3x^2+4y^2)^2$ for all $(x,y) \in \mathbb{R}^2$. [Standard theorems may be used without proof, provided that they are clearly stated.]