Trinity College Dublin, School of Mathematics

MA2342 - Solutions to sample exam.

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Comment: During the real exam, you do not need to write that many explanations. Write down only key thoughts which demonstrate your understanding

1. Hamiltonian Mechanics

1.a.

The Lagrangian of a free particle in three dimensional space is

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Spherical coordinates are given by

$$x = r\cos\varphi\sin\theta, \qquad y = r\sin\varphi\sin\theta \qquad z = r\cos\theta$$

and therefore,

$$\begin{aligned}
\dot{x} &= \dot{r}\cos\varphi\sin\theta - r\dot{\varphi}\sin\varphi\sin\theta + r\dot{\theta}\cos\varphi\cos\theta \\
\dot{y} &= \dot{r}\sin\varphi\sin\theta + r\dot{\varphi}\cos\varphi\sin\theta + r\dot{\theta}\sin\varphi\cos\theta \\
\dot{z} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta.
\end{aligned}$$

Squaring them, and replacing them into the Lagrangian, we get that

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right).$$

If $r(t) = R$ and $\dot{r} = 0$, we get instead

$$\mathcal{L} = \frac{mR^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right).$$

One can notice that since R is a constant, there is indeed only 2 degrees of freedom, θ and φ.

1.b.

The conjugated momenta are found as

$$\begin{aligned} p_\theta &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{\theta}} \\ &= mR^2\dot{\theta} \end{aligned}$$

and

$$\begin{aligned} p_\varphi &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{\varphi}} \\ &= mR^2\dot{\varphi}\sin^2\theta. \end{aligned}$$

Note that this allows us to write

$$\dot{\theta} = \frac{p_\theta}{mR^2}, \qquad \dot{\varphi} = \frac{p_\varphi}{mR^2\sin^2\theta}.$$

The Hamiltonian can be found by the standard procedure. Alternative: to note that technically the Lagrangian looks exactly as the one of two free massive particles: $\mathcal{L} = \frac{m_1}{2}\dot{\theta}^2 + \frac{m_2}{2}\dot{\varphi}^2$, with $m_1 = mR^2$ and $m_2 = mR^2\sin^2\theta$. For such Lagrangian the Legendre transform is well-known: $\mathcal{H} = \frac{p_\theta^2}{2m_1} + \frac{p_\varphi^2}{2m_2}$, therefore the answer:

$$\mathcal{H} = \frac{1}{2mR^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right).$$

The equations of motion are therefore

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{mR^2}, & \dot{\varphi} &= \frac{p_\varphi}{mR^2\sin^2\theta} \\ \dot{p}_\varphi &= 0 & \dot{p}_\theta &= -\frac{\partial\mathcal{H}}{\partial\theta} = \frac{1}{mR^2}p_\varphi^2\frac{\cos\theta}{\sin^3\theta} \end{aligned}$$

1.c.

Recall from 1.a), but restricting ourselves to a sphere, that

$$\begin{aligned} x &= R\cos\varphi\sin\theta \\ y &= R\sin\varphi\sin\theta \\ z &= R\cos\theta \end{aligned}$$

and that the derivatives are, replacing time derivatives with conjugated momenta,

$$\begin{aligned}
\dot{x} &= -\frac{p_\varphi}{mR}\frac{\sin\varphi}{\sin\theta} + \frac{p_\theta}{mR}\cos\varphi\cos\theta \\
\dot{y} &= +\frac{p_\varphi}{mR}\frac{\sin\varphi}{\sin\theta} + \frac{p_\theta}{mR}\sin\varphi\cos\theta \\
\dot{z} &= -\frac{p_\theta}{mR}\sin\theta.
\end{aligned}$$

Therefore, since $M_i = m\epsilon_{ijk}r_j\dot{r}_k$, we get that

$$\begin{aligned}
M_x &= -p_\theta\sin\varphi - p_\varphi\cot\theta\cos\varphi \\
M_y &= -p_\varphi\cot\theta\sin\varphi + p_\theta\cos\varphi \\
M_z &= p_\varphi
\end{aligned}$$

1.d.

Let us first compute $\{M_z, \mathcal{H}\}$. Notice that the only derivative of M_z that doesn't vanish is with respect to p_φ, therefore $\{M_z, \mathcal{H}\} = -\frac{\partial M_z}{\partial p_\varphi}\frac{\partial \mathcal{H}}{\partial \varphi}$. But the Hamiltonian doesn't depend explicitly on φ, therefore this Poisson bracket is 0. Now, the system has spherical symmetry, and the particle is free, therefore no angular momentum direction is prefered, and all the Poisson brackets $\{M_i, \mathcal{H}\} = 0$.

For $\{M_i, M_j\}$, first notice that $\{M_x, M_z\} = \frac{\partial M_x}{\partial \varphi}\frac{\partial M_z}{\partial p_\varphi}$, since the only derivative of M_z that doesn't vanish is with respect to p_φ. Therefore,

$$\{M_x, M_z\} = -p_\theta\cos\varphi + p_\varphi\cot\theta\sin\varphi = -M_y.$$

Again, by symmetry of the system, we can deduce

$$\begin{aligned}
\{M_y, M_z\} &= M_x\,, \\
\{M_x, M_y\} &= M_z.
\end{aligned}$$

Comment: You may encounter Poisson brackets which cannot be computed by the symmetry argument, therefore it is important that you are capable to perform computations of e.g. $\{M_x, M_y\}$ explicitly.

2. Hamilton-Jacobi equation

This section was solved by Adam Keilthy, all credit goes to him.

2.a.

Consider a system with Hamiltonian $\mathcal{H}(q, p, t)$. Let $S_1(q, Q, t)$ be a function which generates a canonical transformation that has the property $\mathcal{H}'(Q, P, t) = 0$.

Comment: An example of such canonical transformation is to take $q(t), p(t)$ and evolve them back, following the reverse of the Hamiltonian flow, to the initial conditions. Then $Q = q(0)$ and $P = p(0)$.

Considering the canonical transformation generated by Hamiltonian flow, we can find, given a Hamiltonian $\mathcal{H}(q, p, t)$, a transformed Hamiltonian $\mathcal{H}'(Q, P, t) = 0$ for all time, therefore such that for all time $\dot{Q} = 0$ and $\dot{P} = 0$.
Generically, $\mathrm{d}S_1 = p\,\mathrm{d}q - P\,\mathrm{d}Q - (\mathcal{H} - \mathcal{H}')\,\mathrm{d}t$, but since $\mathcal{H}' = 0$, one has

$$\mathrm{d}S_1 = p\,\mathrm{d}q - P\,\mathrm{d}Q - \mathcal{H}\,\mathrm{d}t$$

By the definition of the exact differential:

$$\frac{\partial S_1}{\partial t} = -\mathcal{H}$$

and

$$\frac{\partial S_1}{\partial q}|_Q = p$$

Now, we explicitly know $\mathcal{H}(q, p, t)$. We should substitute $p = \frac{\partial S_1}{\partial q}|_Q$ to get the equation:

$$\frac{\partial S_1}{\partial t} + \mathcal{H}(q, \frac{\partial S_1}{\partial q}, t) = 0\,.$$

It is the time-dependent Hamilton-Jacobi equation. It is considered as an equation on $S_1(q)$. Q's play the role of the boundary conditions. In this context, S_1 is often called the Hamilton principal function.
If $\mathcal{H}$ does not depend on time explicitly, we can perform a separation of variables procedure: Consider an ansatz $S_1(q, Q, t) = W(q, Q) - E\,t$ and substitute it to the Hamilton-Jacobi equation. One gets:

$$\mathcal{H}(q, \frac{\partial W}{\partial q}) = E$$

This is called time-independent Hamilton-Jacobi equation. W is called the Hamilton characteristic function.
In generic separation of variables approach, E is assumed to be function of time. In this particular case, we see that the left-hand side of the last equation is time-independent, hence E is time independent, i.e. it is just a constant.

2.b.

We have that

$$W = x\sqrt{2E - \frac{1}{x^2}} + \arctan\left(\frac{1}{x\sqrt{2E - \frac{1}{x^2}}}\right)\,.$$

By the Hamilton-Jacobi equation, we have that

$$E = \mathcal{H}(x, \frac{\partial W}{\partial x})$$

Attempt the Hamiltonian of the standard form, $T + V$:

$$\mathcal{H}(x, \frac{\partial W}{\partial x}) = T + V(x) = \frac{p^2}{2m} + V(x).$$

Let us now consider the fact that $p = \frac{\partial W}{\partial x}$. Computing it, we get

$$\begin{aligned}\frac{\partial}{\partial x} x\sqrt{2E - x^{-2}} &= \sqrt{2E - x^{-2}} + \frac{1}{x^2\sqrt{2E - x^{-2}}} \\ &= \frac{2Ex^2}{x^2\sqrt{2E - x^{-2}}}\end{aligned}$$

and

$$\frac{\partial}{\partial x} \arctan\left(\frac{1}{x\sqrt{2E - \frac{1}{x^2}}}\right) = \frac{-1}{x^2\sqrt{2E - x^{-2}}}$$

Therefore,

$$\frac{\partial W}{\partial x} = \sqrt{2E - x^{-2}}.$$

This yields

$$E = \frac{2E - x^{-2}}{2m} + V(x)$$

and therefore,

$$V(x) = \frac{E(m-1)}{m} + \frac{1}{2mx^2}\,.$$

If you look on the time-independent HJ equation, it is clear that the Hamiltonian cannot depend explicitly on E. Also, this is correct based on common sense. The value of the Hamiltonian is energy, it is not a function of it.
Independence of E is only possible if $m = 1$. Therefore $V = \frac{1}{2x^2}$ and

$$\mathcal{H}(q, p) = \frac{1}{2}\left(p^2 + \frac{1}{x^2}\right)\,.$$

2.c.

We can benefit from knowing

$$\frac{\partial W}{\partial E} = t\,. \tag{2.1}$$

It is enough to know this fact, but here is explanation: $S_1(q, Q, t)$ is a function on a three-dimensional space parameterised by q, Q, t. To each point of this space we can assign not only p, P, but also E (value of energy at a given point). Obviously then,

$dS_1 = pdq - PdQ - Edt$, because Hamiltonian evaluates to energy at each point. Then $dW = d(S_1 + Et) = pdq - PdQ + tdE$, i.e. W is a Legendre transform (with proper signs) of S_1 with respect to time. Another way to get the same result is to note that $S_1(q, Q, t)$ is not an explicit function of E, therefore $\frac{\partial S_1}{\partial E}|_{q,Q,t} = 0$, from where and $S_1 = W - E\,t$ we get (2.1).
Hence we compute

$$t = \frac{\partial W}{\partial E} = \frac{x}{2E}\sqrt{2E - \frac{1}{x^2}}$$

from where we can find x as a function of time:

$$x = \pm\frac{1}{\sqrt{2E}}\sqrt{1 + 4E^2t^2}\,.$$

Dependence on p as a function of time is found from $E = \frac{1}{2}(p^2 + x^{-2})$:

$$p = \pm(2E)^{3/2}\frac{t}{\sqrt{1 + 4E^2t^2}}$$

Dependence on initial conditions can be introduce by replacing $t \to (t - t_0)$. Signs in expression for x and p are correlated. It is clear from physical intuition. Just draw the shape of potential and convince yourself that If $x > 0$, then particle would move to the right when $t \gg 1$.

It is good idea to spend one minute now and check this result. For instance we can verify that equations of motion are satisfied. Then, at $t = 0$ one has $p = 0$ and $x = \pm\frac{1}{\sqrt{2E}}$. When $t \to \infty$, $x = 0$ and $p = \pm\sqrt{2E}$. It is easy to compute energy in both cases, and to check that it is equal to E.

Below is the Adam's solution. The overall logic is correct, but I did not check computation itself for details. Such solution can be highly and probably maximally scored, however it does not take advantage of knowing W. Let us first compute $x(t, E)$. We have that

$$\dot{x} = \frac{\partial H}{\partial p},$$

hence,

$$\dot{x} = \frac{p}{m} = \frac{1}{m}\sqrt{2E - x^{-2}}.$$

Therefore, we need to solve

$$\int \frac{\mathrm{d}t}{m} = \int \frac{x\,\mathrm{d}x}{\sqrt{2Ex^2 - 1}}$$
$$\int \frac{2E\,\mathrm{d}t}{m} = \int \frac{u\,\mathrm{d}x}{\sqrt{u^2 - 1}}$$

with $u = x\sqrt{2E}$.

Let $u = \cosh\theta,\ \mathrm{d}u = \sinh\theta\,\mathrm{d}\theta$ and this becomes, setting $t_0 = 0$,

$$\begin{aligned}\frac{2Et}{m} &= \int \cosh\theta\,\mathrm{d}\theta \\ &= \sinh\theta - \sinh\theta_0 \\ &= \sqrt{u^2-1} - \sqrt{u_0^2-1} \\ &= \sqrt{2Ex^2-1} - \sqrt{2Ex_0^2-1}\end{aligned}$$

which implies

$$x(t,E) = \frac{1}{\sqrt{2E}}\sqrt{1+\left(2Ex_0^2-1+\frac{2Et}{m}\right)^2}$$

and, since we have that $p = \sqrt{2E - x^{-2}}$, we can substitute $x(t,E)$ in to get $p(t,E)$

2.d.

The closest that we can get is when the potential is maximised, hence when $p^2 = 0$. This implies

$$\frac{1}{2mx^2} = E$$

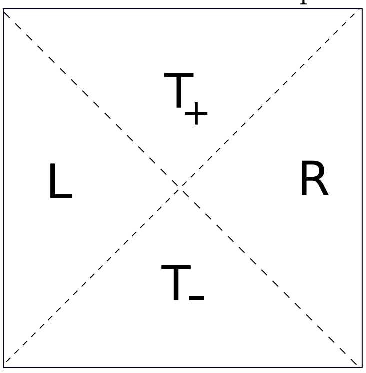
which means that the closest we can get is at

$$|x| = \frac{1}{\sqrt{2E}}\,.$$

3. Special Relativity

3.a.

Lorentz transformation is any linear transformation of space-time coordinates that preserves the value of the interval. Because the interval can have any sign or equal to zero, it is natural to decompose Minkowsky diagram to several zones:



In the zones T_+ (absolute future), T_- (absolute past) the interval is positive (called time-like). In the zones R and L the interval is negative (called space-like). On the diagonals the interval is null (called light-like).

In 1+3 dimensions, R and L are not disconnected regions, but R and L rather refer to the orientation we use to describe the space: right-handed or left-handed.
Lorentz transformations can be separated into four connected components:

1. Preserve the direction of time and the orientation of space. I.e. they preserve the domains of the diagram:

$$T_+ \to T_+ \,, \quad T_- \to T_- \,, \quad R \to R \,, \quad L \to L \,.$$

The collection of these transformations is denoted by SO(1,3) (called Special Lorentz group).

2. Preserve the direction of time, change the orientation of space:

$$T_+ \to T_+ \,, \quad T_- \to T_- \,, \quad R \to L \,, \quad L \to R \,.$$

The basic example of this transform is the so called P-transformation:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{3.1}$$

3. Change the direction of time, preserve the orientation of space:

$$T_+ \to T_- \,, \quad T_- \to T_+ \,, \quad R \to R \,, \quad L \to L \,.$$

The basic example of this transform is the time reversal:

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.2}$$

4. Change the direction of time, change the orientation of space:

$$T_+ \to T_- \,, \quad T_- \to T_+ \,, \quad R \to L \,, \quad L \to R \,.$$

This is done, for instance, by a subsequent application of T and P, denoted by PT.

Consider now any transformation of type 2. Denote it by A. It is clear that $L = P \times A$ belongs to SO(1,3). Indeed, A changes the orientation of space, but P changes it back, so L preserves orientation of space. Since $P^2 = 1$, we can write $A = P \times L$. So any transformation of type 2 can be represented as P times a transformation from SO(1,3), this fact we denote by $P \times \mathrm{SO}(1,3)$.
For the same reason, any transformation of type 3 can be represented as $T \times \mathrm{SO}(1,3)$, and any transformation of type 4 can be represented as $PT \times \mathrm{SO}(1,3)$.

Hence the full Lorentz group has the structure

$$\{1, P, T, PT\} \times \mathrm{SO}(1,3)\,.$$

Comment: If you wrote down the last relation and basic notion of what it is P and T, it would be enough.

Now we should specify what kind of transformations belong to SO(1,3).

First, there are spatial rotations, which do not affect time. They have the structure

$$\begin{pmatrix} 1 & 0\,0\,0 \\ 0 & \\ 0 & \mathcal{O} \\ 0 & \end{pmatrix},$$

where $\mathcal{O}$ is a 3×3 matrix that satisfies $\mathcal{O}\mathcal{O}^T = 1$. The equality $\mathcal{O}\mathcal{O}^T = 1$ follows for instance from condition

$$L_\mu{}^{\mu'} L_\nu{}^{\nu'} \eta_{\mu'\nu'} = \eta_{\mu\nu}$$

applicable for any Lorentz transformation.

Second, there are Lorentz boosts. The Lorentz boost along x-axis is given by

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There also Lorentz boosts along y- and z-axes.

Any special Lorentz transformation can be given by a product of the following 6: 3 Lorentz boosts (along 3 axes), and 3 rotations (around 3 axes).

3.b.

Let $p1^0 = m_1 c\cosh\theta_1$, $p1^1 = m_1 c \sinh\theta_1$, $p2^0 = m_2 c\cosh\theta_2$ and $p2^1 = m_2 c\sinh\theta_2$. Let us apply the Lorentz transform

$$\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$$

to each of them. This yields

$$(p1)' = \Lambda p1 = \begin{pmatrix} m_1 c(\cosh\theta\cosh\theta_1 - \sinh\theta\sinh\theta_1) \\ m_1 c(-\sinh\theta\cosh\theta_1 + \cosh\theta\sinh\theta_1) \end{pmatrix}$$

and similarly

$$(p2)' = \Lambda p2 = \begin{pmatrix} m_2 c(\cosh\theta\cosh\theta_2 - \sinh\theta\sinh\theta_2) \\ m_2 c(-\sinh\theta\cosh\theta_2 + \cosh\theta\sinh\theta_2) \end{pmatrix}.$$

Using the relation $\cosh(\varphi+\psi) = \cosh\varphi\cosh\psi + \sinh\varphi\sinh\psi$, this allows us to write

$$\begin{aligned}
(p1^0)' &= m_1 c\cosh(\theta_1 - \theta) \\
(p2^0)' &= m_2 c\cosh(\theta_2 - \theta) \\
(p1^1)' &= m_1 c\sinh(\theta_1 - \theta) \\
(p2^1)' &= m_2 c\sinh(\theta_2 - \theta)\,.
\end{aligned}$$

Take now $\theta = \theta_1$. Then one gets:

$$\begin{aligned}
&(p1^0)' = m_1 c \\
&(p2^0)' = m_2 c\cosh(\theta_2 - \theta_1) \\
&(p1^1)' = 0 \\
&(p2^1)' = m_2 c\sinh(\theta_2 - \theta_1)\,.
\end{aligned}$$

Since Φ is a scalar, its value does not change from one frame to another. But in the reference frame obtained by boost with $\theta = \theta_1$, the answer depends only on difference of rapidities. Note that the difference of rapidities is itself is a scalar under the Lorentz transformation. Hence Φ will depend only on $(\theta_2 - \theta_1)$ in any reference frame.

3.c.

We discussed quite a lot this question, it is hence skipped here.

4. Particle in an electro-magnetic field

4.a.

Replacing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we get that

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = \partial_\mu\partial_\nu A_\lambda - \partial_\mu\partial_\lambda A_\nu + \partial_\nu\partial_\lambda A_\mu - \partial_\nu\partial_\mu A_\lambda + \partial_\lambda\partial_\mu A_\nu - \partial_\lambda\partial_\nu A_\mu$$

Since partial derivatives commute, this is equal to 0.

4.b.

For this section, I will use the notation

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\theta} = x^{\mu,\theta}, \qquad \frac{\mathrm{d}x_\mu}{\mathrm{d}\theta} = x_{\mu,\theta}$$

The equations of motion for a particle are found by setting $\delta S_{free} + \delta S_{int} = 0$. Let us rewrite the interaction action as

$$\int A_\mu\,\mathrm{d}x^\mu = \int A_\mu x^{\mu,\theta}\,\mathrm{d}\theta$$

and, since the action is the integral of the Lagrangian, the particle Lagrangian is

$$L = -mc\sqrt{\eta_{\mu\nu}x^{\mu,\theta}x^{\nu,\theta}} - \frac{e}{c}A_\mu x^{\mu,\theta}$$

Let us then compute

$$\frac{\partial L}{\partial x^\mu} = -\frac{e}{c}\partial_\mu A_\nu x^{\nu,\theta};$$

$$\frac{\partial L}{\partial x^{\mu,\theta}} = -mc\frac{\eta_{\mu\nu}x^{\nu,\theta}}{\sqrt{\eta_{\mu\nu}x^{\mu,\theta}x^{\nu,\theta}}} - \frac{e}{c}A_\mu.$$

Differentiating the last equation with respect to θ and lowering the index, the Euler-Lagrange equation yields the equations of motion as

$$mc\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\eta_{\mu\nu}x^{\nu,\theta}}{\sqrt{\eta_{\mu\nu}x^{\mu,\theta}x^{\nu,\theta}}} + \frac{e}{c}\partial_\nu A_\mu x^{\nu,\theta} - \frac{e}{c}\partial_\mu A_\nu x^{\nu,\theta} = 0$$

Recalling the definition of $F_{\mu\nu}$ this allows us to write them in the form

$$mc\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\eta_{\mu\nu}x^{\nu,\theta}}{\sqrt{\eta_{\mu\nu}x^{\mu,\theta}x^{\nu,\theta}}} = \frac{e}{c}F_{\mu\nu}x^{\nu,\theta}.$$

For the Maxwell equation, we need to write S_{int} in a continuous fashion. To this end, introduce the current

$$j^\mu = e\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\delta^{(3)}(\mathbf{x} - \mathbf{x}(t)),$$

where $\delta^{(3)}$ is a three-dimensional Dirac Delta function. One can note, through direct integration, that

$$-\frac{e}{c}\int A_\mu(x)\,\mathrm{d}x^\mu = -\frac{1}{c^2}\int \mathrm{d}^4x A_\mu(x)j^\mu(x).$$

The Lagrangian corresponding to the Maxwell equation is

$$\mathcal{L} = -\frac{1}{c^2}A_\mu(x)j^\mu(x) - \frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}.$$

Differentiating with respect to the field A yields

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = -\frac{1}{c^2}j^\nu.$$

As for the second term, we will take care of constants later, let us start with

$$\begin{aligned}\frac{\partial F_{\alpha\beta}F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)} &= F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\\ &= 2F^{\alpha\beta}\frac{\partial(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}{\partial(\partial_\mu A_\nu)}\end{aligned}$$

where we were able to get the second line by lowering indices inside the derivative, which caused indices outside to be raised. Continuing, this yields

$$\begin{aligned}\frac{\partial F_{\alpha\beta}F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)} &= 2F^{\alpha\beta}(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)\\ &= 2F^{\mu\nu} - 2F^{\nu\mu}\\ &= 4F^{\mu\nu}\end{aligned}$$

where the last line is because $F^{\mu\nu}$ is antisymmetric by definition. Therefore, the Euler-Lagrange equation are

$$\frac{\partial_\mu F^{\mu\nu}}{4\pi c} - \frac{1}{c^2}j^\nu = 0,$$

which can be written in the form

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu;$$

that is one of the pairs of Maxwell's equation. The other pair is the one from 4.a)

4.c.

It is clear that the determinant is an invariant since

$$\det(\Lambda^\mu_{\mu'}\Lambda^\nu_{\nu'}F_{\mu\nu}) = \det(\Lambda^\mu_{\mu'})\det(\Lambda^\nu_{\nu'})\det(F_{\mu\nu}) = \det(F_{\mu\nu}).$$

Computing the determinant, one gets that it is equal to

$$\det F = (E^xB^x + E^yB^y + E^zB^z)^2 = (\mathbf{E}\cdot\mathbf{B})^2,$$

therefore $\mathbf{E}\cdot\mathbf{B}$ is an invariant.
It is also clear that $F_{\mu\nu}F^{\mu\nu}$ is an invariant as they transform with inverse Lorentz transformations, that will cancel each other. We can compute that

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B}^2),$$

therefore $\mathbf{E}^2 - \mathbf{B}^2$ is an invariant as well.
The other two are not invariant, $\mathbf{E}^2 + \mathbf{B}^2$ because Lorentz transforms do not preserve three dimensional length, and the norm of the cross product for the same reason.

4.d.

Let us start with the equation of motion

$$mc\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{x_{\mu,\theta}}{\sqrt{\eta_{\alpha\beta}x^{\alpha,\theta}x^{\beta,\theta}}} = \frac{e}{c}F_{\mu\nu}\,x^{\nu,\theta}\,.$$

In the following, put $\theta = ct$. Then $x^{0,ct} = 1$, $x^{i,ct} = \frac{v^i}{c} = \beta^i$, and $\frac{1}{\sqrt{\eta_{\alpha\beta}x^{\alpha,\theta}x^{\beta,\theta}}} = \gamma$.

So the equation of motion becomes:

$$m\frac{\mathrm{d}}{\mathrm{d}t}\gamma\dot{x}_\mu = \frac{e}{c}\,F_{\mu\nu}\dot{x}^\nu\,.$$

Use that $\dot{x}_0 = \dot{x}^0 = c$ and $\dot{x}^i = -\dot{x}_i = v^i$.
In the case $\mu = 0$ the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma mc^2 = e\mathbf{E}\cdot\dot{\mathbf{x}}$$

In the case where $\mu = i$ one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma m\dot{\mathbf{x}} = e(\mathbf{E} - \frac{1}{c}\dot{\mathbf{x}}\times\mathbf{B}).$$

For the Maxwell equations, the one from question 4.a) becomes

$$\nabla\cdot\mathbf{B} = 0 \qquad \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0$$

and the one from question 4.b) becomes

$$\nabla\cdot\mathbf{E} = 4\pi\rho \qquad -\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

This is taken directly from tutorial 7, the derivation in each cases consists in considering cases where $\mu = 0$ separately, and to take the derivatives with respect to ct explicitly, and using the fact that, by definition,

$$j^\mu = (c\rho, \mathbf{J}), \qquad \partial_\mu = (\partial^0, \nabla), \qquad \partial^\mu = (\partial^0, -\nabla)$$