Tutorial 7

Outside class hours, you may ask the lecturer questions in person or by email vel145@gmail.com about this Tutorial or about anything related to the course.

I provide you answers for the questions. Most of the questions were partially or fully discussed on the lecture, check your notes. Please be honest with yourself when performing derivations (in particular, about signs).

Comment: For j^μ I use a different normalisation than the one used on the lecture. I switched so as to use the most common one across the literature.

The action of a particle in electromagnetic field consists of three terms:

$$S = S_{\text{free}} + S_{\text{int}} + S_{EM} \tag{1}$$

The free particle action reads

$$S_{\text{free}} = -m\,c \int \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\theta}\frac{dx^\nu}{d\theta}}\, d\theta\,. \tag{2}$$

1. Derive equation of motion (which is the Euler-Lagrange equation derived from $\delta S_{\text{free}} = 0$).
2. By considering the case $\theta = c\,t$, show that trajectory of a free particle is a straight line.

The interaction action reads

$$S_{\text{int}} = -\frac{e}{c}\int A_\mu(x)\, d\,x^\mu\,. \tag{3}$$

3. Derive equations of motion of the free particle in the presence of electromagnetic field (Euler-Lagrange equations derived from $\delta S_{\text{free}} + \delta S_{\text{int}} = 0$. You should get the answer:

$$m\,c\frac{d}{d\theta}\frac{1}{\sqrt{}}\frac{dx_\mu}{d\theta} = \frac{e}{c}F_{\mu\nu}\frac{dx^\nu}{d\theta}\,, \tag{4}$$

   where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

4. Consider now an explicit realisation of $F_{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}\,. \tag{5}$$

   To avoid confusion, I wrote components of 3-dimensional vectors with an upper subscript, so it is coherent with sign conventions. (e.g. with $A^\mu = \big(A^0, \mathbf{A}\big)$ and $A_\mu = \big(A^0, -\mathbf{A}\big)$).

   Write down explicitly (4) in the 3-dimensional notation, putting $\theta = c\,t$ and considering $\mu = 0$ and $\mu = i$ case separately.

   For the $\mu = 0$ case you should get

$$\frac{d}{dt}(\gamma\, m\, c^2) = e\,\mathbf{E}\cdot\mathbf{v}\,, \tag{6}$$

where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ and γ is the Lorentz factor. Note that $\gamma\, m\, c^2$ is nothing but the energy of the particle.

For the $\mu = i$ case you should get

$$\frac{d}{dt}\left(\gamma\, m\, \mathbf{v}\right) = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right). \tag{7}$$

Note that $\gamma\, m\, \mathbf{v}$ is nothing but the momentum of the particle.

Finally, the electromagnetic action reads (correct your lecture notes: $16\,\pi^2 \to 16\,\pi\, c$):

$$S_{\rm EM} = -\frac{1}{16\,\pi\, c}\int d^4x F_{\mu\nu}F^{\mu\nu}. \tag{8}$$

This last action should be understood as the one for continuous systems as discussed on the last lecture, the Lagrangian density is a functional $\mathcal{L}(\partial_\mu A_\nu)$ In this particular case it depends only through derivatives of A. A generic case would correspond to $\mathcal{L}(A, \partial A)$.

To derive equation of motion, we have to rewrite S_{int} as a 4-dimensional integral. For this we first choose $\theta = c\,t$). Then $S_{int} = -\frac{e}{c}\int A_\mu \frac{dx^\mu}{dt}dt$. Introduce

$$j^\mu(x) = e\frac{dx^\mu}{dt}\delta^{(3)}(\mathbf{x} - \mathbf{x}(\mathbf{t})), \tag{9}$$

so that we can write

$$S_{\rm int} = -\frac{1}{c^2}\int d^4x\, A_\mu(x) j^\mu(x)\,, \tag{10}$$

i.e. also in the style of a continuous system Lagrangian.

5. Derive equations of motion of the free particle in the presence of electromagnetic field (Euler-Lagrange equations for continuous systems derived from $\delta S_{\rm int} + \delta S_{\rm EM} = 0$. Here you should variate with respect to field A, not with respect to trajectory of the particle). The result will be

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu\,, \tag{11}$$

which is one of the pair of Maxwell equations.

6. The second pair of Maxwell equations is $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$. Check that it follows automatically from the fact that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

7. Rewrite (11) in 3-dimensional notations. You should get: $\nabla\cdot\mathbf{E} = 4\pi\,\rho$ and $-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}$. Here by definition $j^\mu = (c\rho, \mathbf{J})$ and the sign convention for derivatives is $\partial_\mu = (\partial^0, \nabla)$, $\partial^\mu = (\partial^0, -\nabla)$.

8. Rewrite $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$ in 3-dimensional notations. You should get $\nabla\cdot\mathbf{B} = 0$ and $\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0$.