Tutorial 6

Outside class hours, you may ask the lecturer questions in person or by email vel145@gmail.com about this Tutorial or about anything related to the course.

Lorentz boost transformation

Below $\beta = v/c$ (it will be called simply velocity), and $\gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}}$. If there is no arrow on top of v and β, then the velocity is considered to be aligned in the x direction (so in practice we are dealing with 1+1 Minkowski space).

Note: It would be very good that you do this section on Tuesday. This is an elementary algebra. In many questions below, you are given exact instructions to follow.

1. Convince yourself that straight lines on Minkowski diagram describe a free particle moving at speed β. How β is related to the slope of the line? Given that particle is moving slower than light, draw correctly relative direction of the light-line and particle line, on a 1+1 Minkowski diagram.

2. Consider the Lorentz boost from the reference frame $\mathcal{O}$ to the reference frame $\mathcal{O}'$:

$$\begin{pmatrix} c\,t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\,t \\ x \end{pmatrix} . \tag{1}$$

   Prove that $(c\,t')^2 - (x')^2 = (c\,t)^2 - x^2$.

3. Derive the formula for relativistic addition of velocities:

$$\beta = \frac{\beta' + \beta''}{1 + \beta'\beta''} . \tag{2}$$

   There are two ways of derivation (do both):

   (a) There is a particle moving at velocity β'' with respect to reference frame $\mathcal{O}'$. This particle can be parameterised by world-line $\{(x^0)' = \tau, (x^1)' = \beta''\tau\}$ (cf. exercise 1). The reference frame $\mathcal{O}'$ is moving with respect to the reference frame $\mathcal{O}$ with velocity β'. Write down the matrix of Lorentz transformation from $\mathcal{O}'$ to $\mathcal{O}$ and act by this matrix on $\{(x^0)', (x^1)'\}$ (i.e. do the inverse operation to (1)). As a result, you will get some explicit parameterisation of the line in $\{x^0, x^1\}$ coordinates, read the velocity β of the particle from this parameterisation. You should get (2). β is interpreted as the speed of the particle with respect to $\mathcal{O}$.

   (b) Consider 3 reference frames: $\mathcal{O}, \mathcal{O}', \mathcal{O}''$. Reference frame $\mathcal{O}'$ is moving with respect to reference frame $\mathcal{O}$ at velocity β'. Reference frame $\mathcal{O}''$ is moving with respect to reference frame $\mathcal{O}'$ at velocity β''. You can make a boost $\mathcal{O} \to \mathcal{O}''$ in two steps: $\mathcal{O} \to \mathcal{O}'$ and $\mathcal{O}' \to \mathcal{O}''$. There are Lorentz transformations L_1 and L_2, a-la (1), corresponding to these boosts. Multiply $L_2 L_1$ (as matrices). The resulting transformation should describe the boost $\mathcal{O} \to \mathcal{O}''$. Read off the β from it. β is interpreted as the speed of $\mathcal{O}''$ with respect to $\mathcal{O}$.

4. From (2), prove that if $|\beta'| < 1$, $|\beta''| < 1$ then $|\beta| < 1$. Meaning: if particle is moving slower than light in one reference frame, then it moves slower than light in any other reference frame. What if say $\beta''| = 1$?

5. Consider the following sum

$$\sum_{i=1}^{M} \gamma(\beta_i)\, m_i\, \beta_i\,.$$

This sum has the meaning of the total momentum of a collection of M particles in a certain reference frame $\mathcal{O}$. Here i labels particles, not coordinates in space.

Prove that if the total momentum is conserved in the reference frame $\mathcal{O}$, then the total momentum in the reference frame $\mathcal{O}'$, moving at velocity β with respect to $\mathcal{O}$, is also conserved (you should use the velocity addition formula and simplify the obtained expressions).

6. Energy and momentum of a relativistic particle of mass m are related by

$$(E/c)^2 - p^2 = m^2\, c^2\,. \tag{3}$$

In 1+1 dimensional case, it is typical to use the rapidity θ parameterisation:

$$E = A\, c^2 \cosh\theta\,, \qquad p = A\, c \sinh\theta\,. \tag{4}$$

Find A.

7. The 2-vector $\{E/c, p\}$ is subject to the transformation (1) when changing from one coordinate system to another. You can check for instance that (3) is invariant under such transformation.

Introduce parameterisation

$$\gamma = \cosh(\theta_\beta)\,, \beta\,\gamma = -\sinh(\theta_\beta)\,, \tag{5}$$

so that the Lorentz boost matrix in (1) becomes

$$\begin{pmatrix} \cosh(\theta_\beta) & \sinh(\theta_\beta) \\ \sinh(\theta_\beta) & \cosh(\theta_\beta) \end{pmatrix} \tag{6}$$

. First, convince yourself that this parameterisation is possible. This is an analog of the 2d rotation of Euclidian space $\left(\begin{smallmatrix} \cos(\phi) & +\sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{smallmatrix}\right)$.

Second, consider the boost from $\mathcal{O}$ to $\mathcal{O}'$ defined by velocity β and hence by rapidity θ_β. Find θ' (rapidity of the particle in the coordinate system $\mathcal{O}'$) as a function of θ (rapidity of the particle in the coordinate system $\mathcal{O}$) and θ_β.

Paradoxes in special relativity

YOU WILL BENEFIT THE MOST FROM THE DISCUSSION OF THE PARADOXES IF YOU THINK ABOUT THEM ON TUESDAY EVENING BEFORE COMING TO THE WEDNESDAY CLASS.

Twin paradox Consider two twins. Twin A makes a journey into a space and back in a high-speed rocket and returns home. Then he finds that the twin B has aged more than he did (as follows from time dilation). But on the other hand, in the reference frame of the twin A, it is the twin B who moved at hight speed. Situation seems symmetrical. Hence it is the twin A who should be more aged then twin B.

Finally, who will be older at the end of the travel?

Ladder paradox The text below is from Wikipedia.

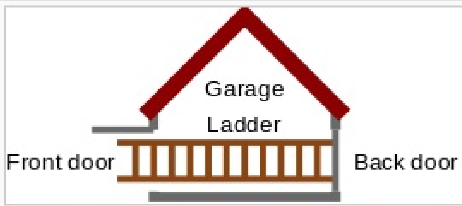

Figure 1: An overview of the garage and the ladder at rest

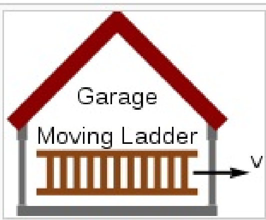

Figure 2: In the garage frame, the ladder undergoes length contraction and will therefore fit into the garage.

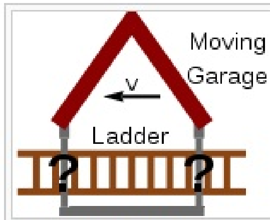

Figure 3: In the ladder frame, the garage undergoes length contraction and is too small to contain the ladder.

The simplest version of the problem involves a garage, with a front and back door which are open, and a ladder which, when at rest with respects to the garage, is too long to fit inside. We now move the ladder at a high horizontal velocity through the stationary garage. Because of its high velocity, the ladder undergoes the relativistic effect of length contraction, and becomes significantly shorter. As a result, as the ladder passes through the garage, it is, for a time, completely contained inside it. We could, if we liked, simultaneously close both doors for a brief time, to demonstrate that the ladder fits.

So far, this is consistent. The apparent paradox comes when we consider the symmetry of the situation. As an observer moving with the ladder is travelling at constant velocity in the inertial reference frame of the garage, this observer also occupies an inertial frame, where, by the principle of relativity, the same laws of physics apply. From this perspective, it is the ladder which is now stationary, and the garage which is moving with high velocity. It is therefore the garage which is length contracted, and we now conclude that it is far too small to have ever fully contained the ladder as it passed through: the ladder does not fit, and we can't close both doors on either side of the ladder without hitting it. This apparent contradiction is the paradox.

Space War Puzzle This one is taken from http://www.einsteins-theory-of-relativity-4engineers.com/space-war.html.

You are observing two fast spaceships flying line astern and distance apart. Exactly when the two ships were equidistant from you, see Figure below, you observed each to fire a single missile simultaneously and dirtily towards each other.

If the speeds of the missiles relative to their respective ships were identical, where, in the inertial frame of the two ships, did that collision happen: (i) equidistant from the two launchers; (ii) closer to the front launcher; (iii) closer to the rear launcher?