Tutorial 5

You are free to use computer (e.g. *Mathematica* or other software) and google the internet.

Outside class hours, you may ask the lecturer questions in person or by email vel145@gmail.com about this Tutorial or about anything related to the course.

Classical Mechanics as the Eikonal limit of a Wave equation

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1. Consider equation

$$\frac{\partial^2\Psi}{\partial t^2} - \sum_{i=1}^{3} A_i \frac{\partial^2\Psi}{\partial x_i^2} = 0\,, \tag{1}$$

where A_i are some positive constants.

Find dispersion relation $\omega = \omega(\mathbf{k})$ for which plane wave ansatz $\Psi = \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})\,t\right)\right]$ solves this equation. For this you have to just plug in the plane wave ansatz to (1) and see what constraint on ω do you get.

Find phase speed and group speed. Show that these two vector objects are not collinear. Find the absolute values of phase speed and group speed. Show that these two values are not equal.

Note: Phase speed is defined as $\mathbf{v}_p \equiv \frac{\omega}{|k|}\mathbf{n}$*, where* $\mathbf{n} = \frac{\mathbf{k}}{|k|}$*. Group speed is defined as* $\mathbf{v}_g \equiv \frac{\partial\omega}{\partial\mathbf{k}}$*. The physical reason for such definitions was discussed during the lecture.*

2. Consider the time-independent Hamilton-Jacobi equation

$$(\nabla W)^2 = 2\mathrm{m}(E - V)\,, \tag{2}$$

and consider the case when $V = \frac{\alpha}{r^2}$.

(a) For the case when $W(r)$ is a function of distance only, solve the Hamilton-Jacobi equation explicitly.
Hint: Since $W(r)$ is a function of r only, you can reduce (2) *to the ordinary differential equation which you know how to solve.*

(b) Solution of the previous question is valid in any number of spatial dimensions. Consider the case of two spatial dimensions. Plot explicitly constant phase (constant W) surfaces for equidistant values of E (e.g. for $E = E_0$, $E = 2E_0$, $E = 3E_0, \ldots$).

(c) Recall that $S = W(x) - E\,t$ is interpreted as a phase of a wave

$$\Psi = \mathcal{A}\, e^{\frac{i}{\hbar}S}\,, \tag{3}$$

and E plays the role of ω, while ∇W plays the roles of $\mathbf{k}$.
Find the phase and the group velocities. Plot the absolute values of these velocities as functions of r.

3. Consider Schrodinger equation:

$$-i\hbar\frac{\partial\Psi}{\partial t} - \frac{\hbar^2}{2m}\Delta\Psi + V(x)\,\Psi = 0\,. \tag{4}$$

Plug in (3) instead of Ψ, apply explicitly differentiations (e.g. $\frac{\partial \Psi}{\partial t} = \left[\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial t} + \left(\frac{i}{\hbar}\frac{\partial S}{\partial t}\right)\right]\Psi$), and derive the following expression

$$\hbar^0\,\Psi\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S\right)^2 + V\right] + i\,\hbar\, e^{\frac{i}{\hbar}S}\left(n_1\frac{\partial \mathcal{A}}{\partial t} + n_2\frac{1}{m}(\nabla\mathcal{A})(\nabla S) + n_3\frac{1}{m}\Delta S\right) - \hbar^2\frac{\Delta\mathcal{A}}{2\,m\,\mathcal{A}} = 0 \tag{5}$$

Note this is not an approximation, this is an exact expression. You should restore numerical factors n_1, n_2, n_3 by yourself.

Approximation comes when we consider $\hbar^0$ term as leading order approximation, it gives us H-J equation on S, and $\hbar^1$ term as the leading order equation on $\mathcal{A}$:

$$\left(n_1\frac{\partial \mathcal{A}}{\partial t} + n_2\frac{1}{m}(\nabla\mathcal{A})(\nabla S) + n_3\frac{1}{m}\Delta S\right) = 0 \tag{6}$$

Substitute $\mathcal{A} = \sqrt{\rho}$, recall that $\nabla S = m\mathbf{v}$, where v is the particle (group) velocity, and rewrite this equation as equation on ρ, simplify the result as much as you can.

What is a typical physical meaning of the obtained equation on ρ? Can you make a suggestion what is the physical meaning of $\mathcal{A}^2 = \rho$ in quantum mechanics?