Tutorial 4

!! Please, first do all non-starred problems in ALL sections and then starred problems.

You are free to use computer (e.g. *Mathematica* or other software) and google the internet.

Outside class hours, you may ask the lecturer questions in person or by email vel145@gmail.com about this Tutorial or about anything related to the course.

Canonical transformations

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1. Show that $P = p$, $Q = q$ is canonical. Find $S_1(q, Q)$, $S_2(q, P)$, $S_3(p, Q)$, $S_4(p, P)$ that generate it.
2. For what α the transformation $P = \alpha\, q$, $Q = -\alpha\, p$ is canonical? Find $S_4(p, P)$ for it.
3. For what θ the transformation $P = \cos(\theta)p + \sin(\theta)q$, $Q = -\sin(\theta)p + \cos(\theta)q$ is canonical? Find $S_1(q, Q)$ for it.
4. Find canonical transformation generated by $S_2(q, P) = q^i P_i - \epsilon_{ijk} a^i q^j P_k$. (the system has 3 q's and 3 p's, summation over the repeated indices is assumed). Consider that a is small, so when expressing P in terms of p, keep only linear in a terms. What is its physical meaning of this transformation? You may want to discuss with me to how find the phsyical meaing.
5. The transformations below are not canonical. Find a simple way to modify them so that they become canonical
$$P = p + q\,, \quad Q = 3q(e^{(q+p)^5} + 1) + p(3e^{(q+p)^5} + 1),$$
$$P = q\,p\,, \quad Q = \log\left(p^{2014} q^{2013}\right).$$
6. Consider a free particle of mass m in dimension D = 1. Compute its action $S(q_f, q_i t_f - t_i) = \int_{q_i,t_i}^{q_f,t_f} \left(p\,\dot{q} - \frac{p^2}{2m}\right) dt$. To do this, you have to solve the Hamiltonian equations of motion and express solution in terms of the data q_f, q_i, t. Then substitute this solution to the integral $\int_{q_i,t_i}^{q_f,t_f} \left(p\,\dot{q} - \frac{p^2}{2m}\right) dt$.

   Consider then the function S as the generating function $S(q, Q, t)$ for the canonical transformation (i.e. replace $q_f \to q$, $q_i \to Q$, $t_f - t_i \to t$. Find the coordinates Q, P as functions of q, p, t and q, p as functions of Q, P, t. Find the Hamiltonian $\mathcal{H}'(Q, P, t)$. Recall that when S depends on the time, one should write
$$p\,dq - P\,dQ - (\mathcal{H} - \mathcal{H}')dt = dS\,. \tag{1}$$
   Werify that S solves the Hamilton-Jacobi equation
$$\frac{\partial S}{\partial t} + \mathcal{H}\left(q, \frac{\partial S}{\partial q}\right) = 0\,. \tag{2}$$
7. Do the same computations as in the previous question but for the case of Harmonic oscillator: $\mathcal{H} = \frac{1}{2}(p^2 + \omega^2\, q^2)$.
8. Prove that $\{Q, P\}_{q,p} = 1$ is equivalent to
$$-\frac{\partial P}{\partial q}_{|Q} = \frac{\partial p}{\partial Q}_{|q}\,. \tag{3}$$
9. How does look a similar equality which is equivalent to $\{Q^i, Q^j\}_{q,p} = 0$?