Tutorial 3

!! Please, first do all non-starred problems in ALL sections and then starred problems.

You are free to use computer (e.g. *Mathematica* or other software) and google the internet. Outside class hours, you may ask the lecturer questions in person or by email vel145@gmail.com about this Tutorial or about anything related to the course.

Vector fields and phase portraits

Draw the phase portraits for the systems given below. Try to make them reasonably realistic. For instance, if there are straight lines, be sure that their slopes are correct, if there are ellipses, make their shape at least qualitatively correct.

Note: you can come and use my computer to check your results against the plots generated by Mathematica.

- n=1 (a) $\dot{x} = +2x$
 - (b) $\dot{x} = -2x$

(c)
$$\dot{x} = (x+1)(x-2)(x-3)^2$$

n=2, linear: System is given by equation $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, where A is a 2 × 2 matrix. Explicitly:

$$\dot{x} = A_{11} x + A_{12} y$$
, $\dot{y} = A_{21} x + A_{22} y$.

Consider the following cases:

(a)
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$
, (b) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, (c) $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, (d) $A = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$.
(e) $A = \begin{pmatrix} +\cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix}$ (understand how picture changes when ϕ changes)

n=2, generic: (a) Hamiltonian system of free particle: $\mathcal{H} = \frac{p^2}{2m}$

- (b) Hamiltonian system of the harmonic oscillator: $\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$
- (c) Hamiltonian system $\mathcal{H} = \frac{p^2}{2m} + V(x)$ with the potential V(x) shown in the figure below: $V(x)^{\dagger}$



It is known that the second derivative of V is non-zero at its extrema.

n=3:* This is a famous example of Lorenz attractor. Using any computer software (don't do this by hand), find numerically solutions of the following equations and make a 3D-plot of them:

$$\begin{aligned} \dot{x} &= \sigma \left(y - x \right), \\ \dot{y} &= x \left(\rho - z \right) - y, \\ \dot{z} &= x y - \beta z. \end{aligned} \tag{1}$$

Here x, y, z are functions of time, and σ, β, ρ are constants. Strange attractor is known to appear for $\sigma = 10$, $\beta = 8/3$, $\rho = 28$. You may use these values, or investigate dependence of the answer on different values of σ, β, ρ .

In the following, assume, if otherwise is not stated, that metric is flat $(g_{ij} = \delta_{ij})$ and that symplectic form is of the canonical type: $\omega = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, the latter is for the basis $\{q^1, q^2, \ldots, q^n, p^1, p^2, \ldots, p^n\}$. Note that gradient and symplectic vector fields are respectively defined as

$$v^{i} = g^{ij}\partial_{i}V, \quad v^{i} = \omega^{ij}\partial_{i}\mathcal{H}, \tag{2}$$

where $g^{ij} \equiv (g^{-1})^{ij}$ and $\omega^{ij} \equiv (\omega^{-1})^{ij}$. V is called the potential of a gradient flow, \mathcal{H} is called the Hamiltonian of a symplectic flow.

1. Which of the following vector fields have a potential and which have a Hamiltonian? Find the potential and Hamiltonian. Note: recall the condition of exactness of a differential form and apply it.

(a) $v = \{x, 1\}$, (b) $v = \{-y, x\}$, (c) $v = \{2, 0, 1, 4\}$ (will the answer depend on the year?) (d) $v = \{x, y, z, t\}$. In these examples the coordinates are either $\{x, y\}$ (for 2d-space) or $\{x, y, z, t\}$ (for 4d-space).

- 2. For a 3-dimensional system, i.e. for a 6-dimensional phase space, compute the 3 symplectic vector flows generated by the angular momentum components M_x, M_y, M_z . Note: you may find it useful to use the Poisson bracket properties to achieve the goal quicker. If your computation is correct, you will see that you can draw it in a collection of 2d-pictures. Do the drawing.
- 3. Classify the structure of all possible critical points for a symplectic flow in 2 dimensions (assuming that matrix A in the linearised vector flow equations is diagonalizable and non-degenerate).
- 4. For 2-dimensional case, when the vector field is at the same of gradient and symplectic type? Start by givining a simple example to convince yourself that it is possible. Then try to write a necessary and sufficient condition.
- 5. Write down a gradient vector flow equations explicitly in polar, spherical, and complex coordinates.