Tutorial 2

!! Please, first do all non-starred problems in ALL sections and then starred problems.

You are free to use computer (e.g. *Mathematica* **or other mathematical software) and google the internet.**

Outside class hours, you may ask the lecturer questions by email vel145@gmail.com about this Tutorial or about anything related to the course.

Hamiltonian and Hamiltonian equations of motion

Summary of the procedure to perform:

a) Consider $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ as an equation for $\dot{q}$. Solve it getting $\dot{q} = \dot{q}(q, p)$,

b) Compute $\mathcal{H} = p\dot{q} - \mathcal{L}(q, \dot{q})$ by replacing $\dot{q}$ with its solution from the previous step,

c) Write down equations of motion

$$\dot{q} = \partial_p \mathcal{H}\,, \quad \text{and} \quad \dot{p} = -\partial_q \mathcal{H}. \tag{1}$$

In the equations below $v \equiv \dot{q}$.

1. Find Hamiltonian, and Hamiltonian equations of motion for $\mathcal{L} = \frac{mv^2}{2} - V(q)$.

2. Consider a free particle in 2 dimensions: $\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$. Rewrite the Lagrangian in polar coordinates $x = r\cos\phi$, $y = r\cos\phi$, then derive Hamiltonian, and Hamiltonian equations of motion in terms of r, ϕ and the corresponding generalised momenta p_r, p_ϕ.

3. Find Hamiltnoian, and Hamiltonian equations of motion for $\mathcal{L} = \frac{1}{2}\sum_{i,j} M_{ij}(q)\dot{q}^i\dot{q}^j - V(q^1, q^2, \ldots, q^n)$.

4.* A particle of unit mass on the sphere is given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,,$$

however the following constraint should be respected: $x^2 + y^2 + z^2 = R^2$.

(a) Rewrite the Lagrangian in the spherical coordinates. This Lagrangian will have only 2 dynamical degrees of freedom: θ and ϕ.

(b) Find conjugated momenta p_ϕ, p_θ and write down the Hamiltonian and Hamiltonian equations of motion.

(c) One should expect that the angular momentum $\mathbf{M} = \mathbf{r} \times \dot{\mathbf{r}}$ is conserved. Rewrite the components of the angular momentum in terms of $\theta, \phi, p_\theta, p_\phi$, then check, using Hamiltonian equations of motion, that M_x, M_y, M_z are indeed conserved quantities.

(d) Compute Poisson brackets $\{M_i, M_j\}$, express the answer in terms of M_x, M_y, M_z again. How to write answer compactly, using Levi-Civita symbol?

(e) One has $M_x, M_y, M_z, \mathcal{H}$ as conserved quantities. In total four. So they should constrain all the time-dependent variables, $\theta, \phi, p_\theta, p_\phi$! Something should be wrong with this observation because it implies that particle is always frozen in some point of the sphere which is obviously wrong. Where is problem? *Hint: compute* $M_x^2 + M_y^2 + M_z^2$

Poisson bracket

For any two functions f, g of generalised coordinates and momenta (q's and p's), Poisson bracket $\{f, g\}$ is defined as

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} \right) . \tag{2}$$

1. Compute $\{q^i, p^j\}$, $\{q^i, q^j\}$, $\{p^i, p^j\}$.
2. Prove $\{f, g\} = -\{g, f\}$ (antisymmetry), $\{f, g_1 + g_2\} = \{f, g_1\} + \{f, g_2\}$ (linearity), $\{f, g_1 g_2\} = g_1\{f, g_2\} + \{f, g_1\}g_2$ (Leibniz rule)
3. Using these properties of the Poisson bracket, compute:
   (a) $\{q_i, \mathcal{H}\}$ and $\{p_i, \mathcal{H}\}$ for the above-derived Hamiltonians.
   (b) $\{x\,p_y - y\,p_x, \frac{p_x^2}{2m}\}$ and $\{x\,p_y - y\,p_x, \frac{p_x^2+p_y^2}{2m}\}$
4. Prove that for any function $f(p, q, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\} . \tag{3}$$

Levi-Civita symbol

In n-dimensional vector space, Levi-Civita symbol is rank n fully antisymmetric tensor ϵ. All its components can be found from the knowledge that $\epsilon_{12\dots n} = 1$.

$n = 2$ (a) write ϵ_{ij} explicitly as a 2×2 matrix.

(b) Show that $v_i w^i = 0$ iff $v_i = \Lambda\,\epsilon_{ij} w^j$, where Λ is some constant.

(c) Simplify: $\sum_i \epsilon_{ij}\epsilon_{ik}$ and $\sum_{i,j} \epsilon_{ij}\epsilon_{ij}$.

(d)* Check that for any 2×2 matrix M with $\det M = 1$ one has: $\epsilon \cdot M \cdot \epsilon = -(M^{-1})^T$, where $\cdot$ is understood as a matrix multiplication.

(e)* Check Plucker identities: $v_i\epsilon_{jk} = v_j\epsilon_{ik} + v_k\epsilon_{ji}$ and $\epsilon_{ij}e_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

$n = 3$ (a) How many non-zero elements are there in ϵ_{ijk}? Write their values explicitly.

(b) Compute $\sum_{i,j,k} \epsilon_{ijk}\epsilon_{ijk}$

(c) Define $\hat{M}_{ij} = \sum_k \epsilon_{ijk} M_k$ for some vector $\mathbf{M} = \{M_x, M_y, M_z\}$. Write down $\hat{M}$ explicitly as a 2×2 matrix.

(d) Compute $\sum_{j,k} \epsilon_{ijk}\hat{M}_{jk}$.
This and previous questions establish a one-to-one correspondence between vectors $\mathbf{M}$ and rank-2 antisymmetric tensors $\hat{M}$ in 3-dimensional space

(e) Show that $\sum_{i,j,k} \epsilon_{ijk}\hat{M}_{ij}N_k = \Lambda\,\mathbf{M} \cdot \mathbf{N}$, find the coefficient of proportionality Λ.

Generic n (a) Compute $\sum \epsilon_{i_1 \dots i_n}\epsilon_{i_1 \dots i_n}$