# Tutorial 2

**Outside class hours, you may ask me questions by email vel145@gmail.com about this Tutorial or about anything related to the course.**

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**Hamiltonian and Hamiltonian equations of motion**

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Summary of the procedure to perform:

a) Consider $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ as a system of equations for $\dot{q}^i$. Solve them getting $\dot{q}^i = \dot{q}^i(q,p)$.

b) Compute $\mathcal{H} = \left(\sum_i p_i \dot{q}^i\right) - \mathcal{L}(q, \dot{q})$ by replacing each $\dot{q}^i$ with its solution from the previous step. Your final result should be a function of $p$'s and $q$'s only.

c) Write down equations of motion

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}. \tag{1}$$

I will do all the indices upper or lower in a correct way. However, this is the case of explicit computation, you know what you are doing, so keeping them upper/lower is not a must.

In the equations below $v \equiv \dot{q}$.

Notation $a \equiv b$ means "equal by definition", another equivalent notation is $a := b$. $a \equiv b$ is not an equation to solve. It is just a statement that the same object can be denoted in two different ways. In LaTeX, $\equiv$ is coded by `\equiv` (short of "equivalent").

1. Find Hamiltonian, and Hamiltonian equations of motion for $\mathcal{L} = \frac{mv^2}{2} - V(q)$.
   **Solution/Hint/Comment:**
   (a) The first step is to find $p$: $p = \frac{\partial \mathcal{L}}{\partial v} = mv$, then we have to express $v$ in terms of $p$: $v = p/m$.
   (b) We compute $\mathcal{H} = p\,v - \mathcal{L}$ and each time when we see $v$, we replace it by $p/m$:

   $$\mathcal{H} = p\,v - \frac{mv^2}{2} + V(q) = p\,\frac{p}{m} - \frac{p^2}{2m} + V(q) = \frac{p^2}{2m} + V(q)\,. \tag{2}$$

   Some of you derived the relation $\mathcal{H} = \frac{mv^2}{2} + V(q)$. This relation is indeed correct, and is used as the first integral of motion of Lagrangian equations. However, this is not what is required here. $v$ is time-derivative of $q$, our goal is to get rid of all time derivatives in the Hamiltonian, so as later be able to write down first-order differential equations on $p$ and $q$.
   (c) Compute equations of motion

   $$\begin{aligned} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}\,, \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial V(q)}{\partial q}\,. \end{aligned} \tag{3}$$

2. Consider a free particle in 2 dimensions: $\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$. Rewrite the Lagrangian in polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, then derive Hamiltonian, and Hamiltonian equations of motion in terms of $r, \phi$ and the corresponding generalised momenta $p_r, p_\phi$.
**Solution/Hint/Comment:**

   (a) One first computes $\dot{x} = -r\sin\phi\,\dot{\phi} + \dot{r}\cos\phi$, $\dot{y} = r\cos\phi\,\dot{\phi} + \dot{r}\sin\phi$, then it is straightforward to compute

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) . \tag{4}$$

   (b) The first step is to express $\dot{r}$ and $\dot{\phi}$ in terms of $p_r$ and $p_\phi$:

$$\begin{aligned} p_r &= \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}\,, \quad \text{so} \quad \dot{r} = p_r/m\,, \\ p_\phi &== \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi}\,, \quad \text{so} \quad \dot{\phi} = p_\phi/mr^2\,. \end{aligned} \tag{5}$$

   (c) Now we compute Hamiltonian:

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2}\,. \tag{6}$$

   Note: there is no need to do computation (b), (c) from scratch. If you look on (4), you notice that $\dot{r}^2$ term is precisely of the form $mv^2/2$, so you know the answer. The term $\dot{\phi}^2$ is also of this form but with $m \to m\,r^2$, so you know the answer as well.

   Note: In this particular case, $p_\phi$ is the angular momentum[1] and $p_r$ is an ordinary momentum (its projection to radial direction). However, in general situation generalised momenta do not necessary have clear physical significance. At first place, they are just suitable parameters that naturally appear in the course of Legendre transformation.

3. Find Hamiltnoian, and Hamiltonian equations of motion for $\mathcal{L} = \frac{1}{2}\sum_{i,j} M_{ij}(q)\dot{q}^i\dot{q}^j - V(q^1, q^2, \ldots, q^n)$.
**Solution/Hint/Comment:** Since $\sum_{i,j} M_{ij}v^iv^j = \sum_{i,j} M_{ji}v^iv^j = \sum_{i,j}\frac{M_{ij}+M_{ji}}{2}$, one can consider that $M_{ij} = M_{ji}$. Otherwise, one should replace $M \to (M + M^T)/2$ in the formulae below.

   (a) A typical mistake that was made is $\frac{\partial}{\partial\dot{q}^i}\sum M_{ij}\dot{q}^i\dot{q}^j = \sum M_{ij}\dot{q}^i\dot{q}^j$. Under summation, $i$ is mute summation index, we can give it any other name. It has nothing to do with $i$ in $\frac{\partial}{\partial\dot{q}^i}$. Correct way to procede is to use the following relation:

$$\frac{\partial x^a}{\partial x^b} = \delta^a{}_b \tag{7}$$

   and not to use the same letter for the summation index and index in $\frac{\partial}{\partial\dot{q}^i}$.

$$p_k = \frac{\partial\mathcal{L}}{\partial\dot{q}^k} = \frac{1}{2}\sum_{ij} M_{ij}\frac{\partial}{\partial\dot{q}^k}\left(\dot{q}^i\dot{q}^j\right) = \frac{1}{2}\sum_{ij} M_{ij}\left(\delta_k{}^i\dot{q}^j + \delta_k^j\dot{q}^i\right) = \frac{1}{2}\sum_j M_{kj}\dot{q}^j + \frac{1}{2}\sum_i M_{ik}\dot{q}^i = \sum_j M_{kj}\dot{q}^j$$

[1] check this fact! The angular momentum is the one in $z$-direction (perpendicular to the plane where the system lives): $M_z = x\,p_y - y\,p_x$

In index free notation[2]

$$\mathbf{p} = \hat{M} \cdot \dot{\mathbf{q}},$$

So

$$\dot{\mathbf{q}} = \hat{M}^{-1} \cdot \mathbf{p}\,.$$

(b) We are continuing with index-free notation now (note, $\hat{M}^T = \hat{M}$):

$$\mathcal{H} = \mathbf{p}^T \cdot \dot{\mathbf{q}} - \mathcal{L} = \mathbf{p}^T \cdot \dot{\mathbf{q}} - \frac{1}{2}\dot{\mathbf{q}}^T \cdot \hat{M} \cdot \dot{\mathbf{q}} + V = \frac{1}{2}\,\mathbf{p}^T \cdot \hat{M}^{-1} \cdot \mathbf{p} + V\,.$$

A basic check to do: when $M$ is diagonal, this corresponds to the $\mathcal{L} = \sum_i \frac{m_i}{2} v_i^2 - V$ Lagrangian, you know already the answer for this special case.

(c)

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} = \sum_j \left(M^{-1}\right)^{ij} p_j\,, \qquad \dot{p}_j = -\frac{\partial V}{\partial q^j}\,. \tag{8}$$

4.* A particle of unit mass on the sphere is given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,,$$

however the following constraint should be respected: $x^2 + y^2 + z^2 = R^2$.

(a) Rewrite the Lagrangian in the spherical coordinates. This Lagrangian will have only 2 dynamical degrees of freedom: $\theta$ and $\phi$.

(b) Find conjugated momenta $p_\phi, p_\theta$ and write down the Hamiltonian and Hamiltonian equations of motion.

(c) One should expect that the angular momentum $\mathbf{M} = \mathbf{r} \times \dot{\mathbf{r}}$ is conserved. Rewrite the components of the angular momentum in terms of $\theta, \phi, p_\theta, p_\phi$, then check, using Hamiltonian equations of motion, that $M_x, M_y, M_z$ are indeed conserved quantities.

(d) Compute Poisson brackets $\{M_i, M_j\}$, express the answer in terms of $M_x, M_y, M_z$ again. How to write answer compactly, using Levi-Civita symbol?

(e) One has $M_x, M_y, M_z, \mathcal{H}$ as conserved quantities. In total four. So they should constrain all the time-dependent variables, $\theta, \phi, p_\theta, p_\phi$! Something should be wrong with this observation because it implies that particle is always frozen in some point of the sphere which is obviously wrong. Where is problem? *Hint: compute* $M_x^2 + M_y^2 + M_z^2$

**Solution/Hint/Comment:** Spherical coordinates are $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$. In this problem $R$ should be a constant.

(a) $\mathcal{L} = \frac{R^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$,

[2] $\hat{M}$ is a matrix, $\dot{\mathbf{q}}$ and $\dot{\mathbf{p}}$ are the vector-columns.

(b) $p_\theta = R^2\dot\theta \;\; p_\phi = R^2\sin^2\theta\dot\phi\,,\; \mathcal{H} = \frac{1}{2R^2}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\phi^2\right)$. Equations of motion:

$$\dot\theta = \frac{p_\theta}{R^2}\,,\quad \dot\phi = \frac{p_\phi}{R^2\sin^2\theta}\,,\quad \dot p_\theta = \frac{p_\phi^2}{R^2}\frac{\cos\theta}{\sin^3\theta}\,,\quad \dot p_\phi = 0\,. \tag{9}$$

(c) $M_x = y\,\dot z - z\,\dot y = -\cos\phi\,\cot\theta\,p_\phi - \sin\phi\,p_\theta$
$M_y = z\,\dot x - x\,\dot z = -\sin\phi\,\cot\theta\,p_\phi + \cos\phi\,p_\theta$
$M_z = x\,\dot y - y\,\dot z = p_\phi$. The fact that they are conserved is verified by explicit computation.

(d) $\{M_i, M_j\} = \epsilon_{ijk}M_k$.

(e) A straightforward computations gives us

$$M_x^2 + M_y^2 + M_z^2 = 2R^2\,\mathcal{H}\,. \tag{10}$$

So, only three of four integral of motion are independent.

A nice way to perform the same computation is to note that for a particle on a sphere $\mathbf{r}\cdot\dot{\mathbf{r}} = 0$. This property follows from differentiation of $\mathbf{r}\cdot\mathbf{r} = R^2$. Hence

$$\mathbf{M}^2 = (\mathbf{r}\times\dot{\mathbf{r}})\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = \mathbf{r}^2\,\dot{\mathbf{r}}^2 - (\mathbf{r}\cdot\dot{\mathbf{r}})^2 = R^2\,\dot{\mathbf{r}}^2 = 2R^2\mathcal{L} = 2R^2\mathcal{H}\,. \tag{11}$$

$\mathcal{L} = \mathcal{H}$ is a nice property of this system (it comes from the fact that $\mathcal{L}$ is quadratic form in the velocity vectors).

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## Poisson bracket

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For any two functions $f, g$ of generalised coordinates and momenta ($q$'s and $p$'s), Poisson bracket $\{f,g\}$ is defined as

$$\{f,g\} = \sum_i\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right)\,. \tag{12}$$

Comment: I put position of indices correctly, but this is not absolutely necessary in such kind of explicit computations.

1. Compute $\{q^i, p_j\}$, $\{q^i, q^j\}$, $\{p_i, p_j\}$.
**Solution/Hint/Comment:** These three quantities are computed from explicit definition of the bracket. Answer is

$$\{q^i, p_j\} = \delta^i{}_j\,,\quad \{q^i, q^j\} = 0\,,\quad \{p^i, p^j\} = 0\,. \tag{13}$$

2. Prove $\{f,g\} = -\{g,f\}$ (antisymmetry), $\{f, g_1 + g_2\} = \{f, g_1\} + \{f, g_2\}$ (linearity), $\{f, g_1g_2\} = g_1\{f, g_2\} + \{f, g_1\}g_2$ (Leibniz rule)
**Solution/Hint/Comment:** Derived explicitly from the definition. Leibniz rule discloses for us that Possion bracket $\{f,g\}$ can be thought as: a differential operator constructed from $f$ that acts on $g$:

$$\{f,g\} = \sum_i\left(\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p^i}\frac{\partial}{\partial q_i}\right)g \equiv D_f\cdot g\,, \tag{14}$$

so the Leibniz rule is understood in standard way: $\mathrm{D}_f \cdot (g_1 g_2) = (D_f \cdot g_1) g_2 + g_1 (D_f \cdot g_2)$.

One can add here that the Jacobi identity is a variation of the Leibniz rule. Indeed, it can be written in the form

$$\begin{aligned}\{f,\{g,h\}\} = \\ D_f \cdot \{g,h\} = \{D_f \cdot f, h\} + \{g, D_f \cdot h\} \\ = \{\{f,g\},h\} + \{g,\{f,h\}\}\end{aligned} \tag{15}$$

Note, it requires a bit of work to derive $D_f \cdot \{g,h\} = \{D_f \cdot f, h\} + \{g, D_f \cdot h\}$.

3. Using these properties of the Poisson bracket, compute:

   (a) $\{q_i, \mathcal{H}\}$ and $\{p_i, \mathcal{H}\}$ for the above-derived Hamiltonians.

   (b) $\{x\, p_y - y\, p_x, \frac{p_x^2}{2m}\}$ and $\{x\, p_y - y\, p_x, \frac{p_x^2 + p_y^2}{2m}\}$

   **Solution/Hint/Comment:**

   (a) Note that $\{q^i, f\} = \frac{\partial f}{\partial p_i}$ and $\{p_i, f\} = -\frac{\partial f}{\partial q^i}$. So $\{q^i, \mathcal{H}\}$ and $\{p_i, \mathcal{H}\}$ produce just the r.h.s. of the equations of motion. Equations of motion can be also written in the form

   $$\dot{q}^i = \{q^i, \mathcal{H}\}\,, \qquad \dot{p}_i = \{p_i, \mathcal{H}\}\,. \tag{16}$$

   (b)

   $$\{x\, p_y - y\, p_x, \frac{p_x^2}{2m}\} = \frac{p_x}{m}\{x\, p_y - y\, p_x, p_x\} = \frac{p_x}{m} \times \Big(\{x\, p_y, p_x\} - \{y\, p_x, p_x\}\Big) = \frac{p_x}{m}\,(p_y + 0) = \frac{p_x p_y}{m} \tag{17}$$

   Note $x\, p_y - y\, p_x$ is an angular momentum. Poisson bracket with angular momentum generates rotations. Here we see that $\{x\, p_y - y\, p_x, p_x\} = p_y$. So indeed $p_x$ is rotated up to $p_y$.
   Since $p_x^2 + p_y^2$ is the norm of vector $\mathbf{p}$, it is invariant by rotations. So we expect that

   $$\{x\, p_y - y\, p_x, \frac{p_x^2 + p_y^2}{2m}\} = 0\,. \tag{18}$$

   This is indeed true, can be confirmed by explicit computation in full analogy with (17).

4. Prove that for any function $f(p, q, t)$

   $$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\}\,. \tag{19}$$

   **Solution/Hint/Comment:** For simplicity of notation, consider a single degree of freedom (for many degrees, you have to just add sums):

   $$\frac{df(p,q,t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial t} + \left(\frac{\partial f}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial \mathcal{H}}{\partial q}\right) = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\}\,. \tag{20}$$

**Levi-Civita symbol**

In $n$-dimensional vector space, Levi-Civita symbol is rank $n$ fully antisymmetric tensor $\epsilon$. All its components can be found from the knowledge that $\epsilon_{12\ldots n} = 1$.

$n = 2$

(a) write $\epsilon_{ij}$ explicitly as a $2 \times 2$ matrix.

(b) Show that $v_i w^i = 0$ iff $v_i = \Lambda\, \epsilon_{ij} w^j$, where $\Lambda$ is some constant.

(c) Simplify: $\sum_i \epsilon_{ij}\epsilon_{ik}$ and $\sum_{i,j} \epsilon_{ij}\epsilon_{ij}$.

(d)* Check that for any $2 \times 2$ matrix $M$ with $\det M = 1$ one has: $\epsilon \cdot M \cdot \epsilon = -(M^{-1})^T$, where $\cdot$ is understood as a matrix multiplication.

(e)* Check Plucker identities: $v_i \epsilon_{jk} = v_j \epsilon_{ik} + v_k \epsilon_{ji}$ and $\epsilon_{ij} e_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.

$n = 3$

(a) How many non-zero elements are there in $\epsilon_{ijk}$? Write their values explicitly.

(b) Compute $\sum_{i,j,k} \epsilon_{ijk}\epsilon_{ijk}$

(c) Define $\hat{M}_{ij} = \sum_k \epsilon_{ijk} M_k$ for some vector $\mathbf{M} = \{M_x, M_y, M_z\}$. Write down $\hat{M}$ explicitly as a $2 \times 2$ matrix.

(d) Compute $\sum_{j,k} \epsilon_{ijk} \hat{M}_{jk}$.
This and previous questions establish a one-to-one correspondence between vectors $\mathbf{M}$ and rank-2 anti-symmetric tensors $\hat{M}$ in 3-dimensional space

(e) Show that $\sum_{i,j,k} \epsilon_{ijk} \hat{M}_{ij} N_k = \Lambda\, \mathbf{M} \cdot \mathbf{N}$, find the coefficient of proportionality $\Lambda$.

Generic $n$

(a) Compute $\sum \epsilon_{i_1 \ldots i_n} \epsilon_{i_1 \ldots i_n}$