## Tutorial 1

!! Please, first do all non-starred problems in ALL sections and then starred problems.

You are free to use computer (e.g. Mathematica or other mathematical software) and google the internet.

Outside class hours, you may ask the lecturer questions by email vel145@gmail.com about this Tutorial or about anything related to the course.

## Dual vector space

1. $v_{1}, v_{2} \in V, f_{1}, f_{2} \in V^{*}$. Suppose that in a certain basis of $V v_{1}=\binom{1}{0}$, $v_{2}=\binom{0}{1}$ and in the dual basis of $V^{*}$ (which is uniquely defined by the choice of basis of $V) f_{1}=\left(\begin{array}{ll}1 & 0\end{array}\right), f_{2}=\left(\begin{array}{ll}0 & 1\end{array}\right)$.
In some new basis $v_{1}=\binom{1}{1}, v_{2}=\binom{1}{-1}$. What is the form of $f_{1}, f_{2}$ in the corresponding dual basis?
2. In a certain basis $v_{1}=\binom{1}{0}$ and $f_{1}=\left(\begin{array}{ll}0 & 1\end{array}\right)$. In another basis $v_{1}=\binom{0}{1}$. What are the options for $f_{1}$ ?
3. Consider a 2-dimensional metric space. For certain two basis vectors $\alpha_{1}$ and $\alpha_{2}$ the metric $A_{i j} \equiv\left\langle\alpha_{i}, \alpha_{j}\right\rangle$ is given by $A=\left(\begin{array}{cc}2 & -1 \\ -1 & 2\end{array}\right)$. Here $\langle\cdot, \cdot\rangle$ means scalar product. Express the vectors $\delta^{1}$ and $\delta^{2}$ of the canonically dual (to $\left\{\alpha_{1}, \alpha_{2}\right\}$ ) basis as linear combinations of $\alpha_{1}, \alpha_{2}$.
Recall that presence of metrics allows us to identify $V$ and $V^{*}$, so the question is meaningful.
4. The question is the same as above, but for $n$-dimensional space. Now $A$ is an $n \times n$ dimensional matrix:

$$
A_{i j}=\left(\begin{array}{cccccc}
2 & -1 & 0 & 0 & \ldots & 0  \tag{1}\\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & -1 & 2 & -1 \\
0 & \ldots & 0 & 0 & -1 & 2
\end{array}\right)
$$

It is suggested to understand the answer for the case $n=3,4$ first.

Last two questions have practical application. $\alpha$ 's are the so called simple roots of $\mathfrak{s l}(n+1)$ Lie algebra. A is Cartan matrix. You will encounter them in your study later.

## Vector fields and coordinate transformations

Polar coordinates $r, \phi$ are defined by $x=r \cos \phi, y=r \sin \phi$.
Complex coordinates are defined by $z=x+i y, \bar{z}=x-i y$. In the questions below do not worry that $i$ is not real. You can formally think that $z, \bar{z}$ parameterise 2-dimensional real space (they indeed do, even though you cannot draw coordinate axes for them) and treat $i=\sqrt{-1}$ as just a number that you know how to operate with.

1. Rewrite $x d y-y d x$ in polar coordinates and complex coordinates.
2. For $\omega=y d x+d y$, find such coordinate system $\{u, v\}$ in which $\omega=h(u) d v$, where $h$ is some function.

3* Explain, by counting the number of equations and variables, why for any 1-form $\omega$ in 2 dimensions one can always find a coordinate system such that $\omega=h(u) d v$. Show that a similar statement does not hold in higher dimensions.

Take a slightly different point of view on the topic of the last lecture. To each point $x$ of certain domain D , which can be $\mathbb{R}^{n}$ or its part, we are assigning some object.

A very simple object is a real constant. Such assignment is nothing but definition of a function $f(x)$.

More complicated object is a $k$-dimensional vector. Assigning a vector to each point defines for us a vector field. In practice, we are introducing $k$ functions $\left\{\omega_{1}(x), \ldots, \omega_{k}(x)\right\}$ to describe it, however a nontrivial point is that these functions may depend on the choice of the coordinate system in D. For the case $k=n$ and $\omega_{i}$ being the components of the differential form, we discussed this dependence on the lecture. Remind that differential form is parameterised as follows:

$$
\begin{equation*}
\omega(x)=\omega_{i}(x) d x^{i} \tag{2}
\end{equation*}
$$

Note aside: Take a look one more time on the function $f(x)$ that was discussed above. To precise that the value of the function $f$ at point $x$ does not depend on the choice of a coordinate system, we say that $f$ is a scalar field.

Another example of non-trivial vector structure is tangent vector field. In this case also $k=n$, and the tangent vector field is parameterised as

$$
\begin{equation*}
v(x)=v^{i}(x) \partial_{i} \tag{3}
\end{equation*}
$$

Above, $d x^{i}$ and $\partial_{i}$ can be thought simply as suitable mnemonic notations for basis forms, reps. tangent vectors, at each point $x$.
$v$ at point $x$ is the same as the displacement vector $\Delta \vec{x}$ used during lecture. $\partial_{i}$ can be thought as unit displacements $\Delta x_{i}$ in the basis directions.

Both differential form and tangent vector field are vector fields, but of different nature. At each point $x, \omega(x)$ and $v(x)$ are elements of vector spaces which are dual to one another.
4. Knowing that $d x^{i}$ and $\partial_{i}$ are canonically dual bases, i.e. that $d x^{i}\left(\partial_{j}\right)=\delta_{j}^{i}$, find transformation rule for $v^{i}(x)$ under the change of basis.
5. Show that $\left[{ }_{v} w\right](x) \equiv v^{i}(x) \omega_{i}(x)$ is a scalar field.
6. Consider a set of trajectories of a particle given by $x^{i}\left(t, \theta_{1}, \theta_{2}, \ldots, \theta_{n-1}\right)$, where $t$ is time and $c$ 's are "initial conditions", so that this set covers D.
Example: $x^{1}=t \cos \theta_{1}, x^{2}=t \sin \theta_{1}, \mathrm{D}$ is $\mathbb{R}^{2}$ without origin.
a) Show that the velocity vector of the particle $\left\{\frac{d x^{1}}{d t}, \ldots, \frac{d x^{n}}{d t}\right\}$ defines for us tangent vector field, i.e. that it properly transforms under the change of coordinates (hence the name "tangent vector field" because velocities are vectors tangent to trajectory).
b) Compute the velocity field for the above-mentioned example in the coordinate system $\left\{x^{1}, x^{2}\right\}$ and in the polar coordinates. Draw a plot of this vector field.
7. Rewrite $x \partial_{y}-y \partial_{x}$ in polar coordinates and complex coordinates.

8* Rewrite $x \partial_{y}-y \partial_{x}, x \partial_{z}-z \partial_{x}, y \partial_{z}-z \partial_{y}$ in spherical coordinates. Spherical coordinates are defined by $x=r \sin \theta \cos \phi, y=r \sin \theta \sin \phi, z=r \cos \theta$.
9* Metric is a tensor field of rank $2, g_{i j}(x)$, which defines a scalar product of vectors (at each point $x):\langle v, w\rangle \equiv v^{i} w^{j} g_{i j}$. This is another example of vector field, now with $k=2 n$.
Find how metric changes with the change of coordinates
$10 *$ If in the Descartes coordinates $g_{i j}=\delta_{i j}$, find $g$ explicitly in polar and complex coordinates for the case of 2 -dim space and in spherical coordinates for the case of 3-dim space.

## Integration

1. Is it correct, in general, to write $\int_{\gamma} y d x=y x$ ?
2. Compute integral $\oint_{\gamma} p d q$, where $\gamma$ is a circle of radius $R: p^{2}+q^{2}=R^{2}$. Integration is counterclockwise.
3. Compute integral $\int_{\gamma} d(x / y)$ for the following 3 contours. Each contour consists of straight lines connecting the following points:
a) $\{0,0\},\{1,1\}$,
b) $\{0,0\},\{1,0\},\{1,1\}$,
c) $\{0,0\},\{0,1\},\{1,1\}$.

## Legendre transform

Legendre transform of function $f(x)$ on the interval $I$ is a function $g(p)$ defined as

$$
\begin{equation*}
g(p)=\sup _{x \in I}(x p-f(x)) \tag{4}
\end{equation*}
$$

1. It is well-defined operation if $\frac{\partial^{2} f(x)}{\partial x^{2}} \geq 0$. Why?
2. Find the Legendre transform of $\sqrt{1+x^{2}}$. What is the range of $x$ and $p$ for which the Legendre transform is defined?

## On exact differentials

1. Find all $f$ such that $d f=2 x y d x+\left(x^{2}-y^{2}\right) d y$.
2. Prove that if $\omega$ is exact then $\oint_{\gamma} \omega=0$ for any closed contour.

3* Since for exact differential $\omega$ one has $\omega_{i}=\frac{\partial f}{\partial x^{i}}$, the necessary condition of exactness is

$$
\begin{equation*}
\frac{\partial \omega_{i}}{\partial x^{j}}=\frac{\partial \omega_{j}}{\partial x^{i}} . \tag{5}
\end{equation*}
$$

Prove that if holds than $\oint_{\gamma} \omega=0$ for any closed contour $\gamma$ (note that (5) was not proven to be sufficient condition). Consider for this first $\gamma$ being a square. Then use the argument that any closed contour is a limit of many squares (at this second step you are not required to be rigorous).

4* Consider $\omega=\frac{x d y-y d x}{x^{2}+y^{2}}$. Is condition (5) satisfied? Compute $\int_{\gamma} \omega$ for contour being a circle of unit radius $x^{2}+y^{2}=1$. Do you get zero? Are you happy with the statements that you proved above?

