Tutorial 1 - SOLUTIONS

Dual vector space

1. $v_1, v_2 \in V$, $f_1, f_2 \in V^*$. Suppose that in a certain basis of V $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and in the dual basis of V^* (which is uniquely defined by the choice of basis of V) $f_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

   In some new basis $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. What is the form of f_1, f_2 in the corresponding dual basis?

   Solution/Hint/Comment: If $v \to M \cdot v$ then $f \to f \cdot M^{-1}$. Need to find M. Another option is to solve $f_i(v_j) = \delta_{ij}$ which is a basis independent relation. Answer: $f_1 = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$, $f_2 = \begin{pmatrix} 1/2 & -1/2 \end{pmatrix}$.

2. In a certain basis $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. In another basis $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What are the options for f_1?
   Solution/Hint/Comment: $f_1(v_1) = 0$. This property does not depend on basis. Hence, generically $f_1 = \begin{pmatrix} \lambda & 0 \end{pmatrix}$.

3. Consider a 2-dimensional metric space. For certain two basis vectors α_1 and α_2 the metric $A_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$ is given by $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Here $\langle \cdot, \cdot \rangle$ means scalar product. Express the vectors δ^1 and δ^2 of the canonically dual (to $\{\alpha_1, \alpha_2\}$) basis as linear combinations of α_1, α_2.

   Recall that presence of metrics allows us to identify V and V^*, so the question is meaningful.

   Solution/Hint/Comment: Operation $\langle v, \cdot \rangle$ makes vector v a linear functional on V, hence the member of V^*. This point was discussed on the lecture.

   The fastest way: Consider the ansatz $\delta^i = c^{ij}\alpha_j$ and make a trial and error search for c's so as to satisfy $\langle \delta^i, \alpha_j \rangle = \delta_{ij}$ (which is a property defining the dual basis). Systematic way: $c = A^{-1}$ (easy to prove), so we have to inverse A. The answer is $A^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

4.* The question is the same as above, but for n-dimensional space. Now A is an $n \times n$ dimensional matrix:

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}. \quad (1)$$

It is suggested to understand the answer for the case $n = 3, 4$ first.

Last two questions have practical application. α's are the so called simple roots of $\mathfrak{sl}(n+1)$ Lie algebra. A is Cartan matrix. You will encounter them in your study later.

Solution/Hint/Comment: $(A^{-1})^{ij} = \frac{j(n+1-i)}{n+1}$ for $i \leq j$ and $(A^{-1})^{ij} = (A^{-1})^{ji}$ for $i > j$. By making trial and error game for $n = 3, 4$ one can guess this answer.

Vector fields and coordinate transformations

Polar coordinates r, ϕ are defined by $x = r\cos\phi$, $y = r\sin\phi$.

Complex coordinates are defined by $z = x + i\,y$, $\bar{z} = x - i\,y$. In the questions below do not worry that i is not real. You can formally think that $z, \bar{z}$ parameterise 2-dimensional real space (they indeed do, even though you cannot draw coordinate axes for them) and treat $i = \sqrt{-1}$ as just a number that you know how to operate with.

1. Rewrite $x\,dy - y\,dx$ in polar coordinates and complex coordinates.
   Solution/Hint/Comment: Answer: $r^2\,d\phi$ and $\frac{i}{2}\left(z\,d\bar{z} - \bar{z}\,dz\right)$.

2. For $\omega = y\,dx + dy$, find such coordinate system $\{u, v\}$ in which $\omega = h(u)dv$, where h is some function.
   Solution/Hint/Comment: $v = x + \log y$, $h(u) = u = y$.

3.* Explain, by counting the number of equations and variables, why for any 1-form ω in 2 dimensions one can always find a coordinate system such that $\omega = h(u)dv$. Show that a similar statement does not hold in higher dimensions.
   Solution/Hint/Comment: $h(u)dv = h\frac{\partial v}{\partial x}dx + h\frac{\partial v}{\partial y}dy$. We have two equations, $h\frac{\partial v}{\partial x} = \omega_x$ and $h\frac{\partial v}{\partial y} = \omega_y$ to find two functions. In higher dimensions, there will be three or more equations on two functions.

Take a slightly different point of view on the topic of the last lecture. To each point x of certain domain D, which can be $\mathbb{R}^n$ or its part, we are assigning some object.

A very simple object is a real constant. Such assignment is nothing but definition of a function $f(x)$.

More complicated object is a k-dimensional vector. Assigning a vector to each point defines for us a vector field. In practice, we are introducing k functions $\{\omega_1(x), \ldots, \omega_k(x)\}$ to describe it, however a nontrivial point is that these functions may depend on the choice of the coordinate system in D. For the case $k = n$ and ω_i being the components of the **differential form**, we discussed this dependence on the lecture. Remind that differential form is parameterised as follows:

$$\omega(x) = \omega_i(x)\, dx^i\,. \tag{2}$$

Note aside: Take a look one more time on the function $f(x)$ that was discussed above. To precise that the value of the function f at point x does not depend on the choice of a coordinate system, we say that f is a **scalar field**.

Another example of non-trivial vector structure is **tangent vector field**. In this case also $k = n$, and the tangent vector field is parameterised as

$$v(x) = v^i(x)\partial_i\,. \tag{3}$$

Above, dx^i and ∂_i can be thought simply as suitable mnemonic notations for basis forms, reps. tangent vectors, at each point x.

v at point x is the same as the displacement vector $\Delta\overrightarrow{x}$ used during lecture. ∂_i can be thought as unit displacements Δx_i in the basis directions.

Both differential form and tangent vector field are vector fields, but of different nature. At each point x, $\omega(x)$ and $v(x)$ are elements of vector spaces which are dual to one another.

4. Knowing that dx^i and ∂_i are canonically dual bases, i.e. that $dx^i(\partial_j) = \delta^i_j$, find transformation rule for $v^i(x)$ under the change of basis.
   Solution/Hint/Comment: Below $\mathbf{x} = \{\mathbf{x^1}, \mathbf{x^2}, \ldots, \mathbf{x^n}\}$ and $\mathbf{y} = \{\mathbf{y^1}, \mathbf{y^2}, \ldots, \mathbf{y^n}\}$. These are the coordinates of THE SAME point, but in different coordinate systems.

$$\begin{aligned} \omega_i(x) &= \frac{\partial y^j}{\partial x^i}\omega_j(y) \quad \text{(Discussed on the lecture)} \\ v^i(x) &= \frac{\partial x^i}{\partial y^j} v^j(y) \quad \text{(Required answer)} \end{aligned} \tag{4}$$

5. Show that $[\iota_v w](x) \equiv v^i(x)\omega_i(x)$ is a scalar field.
   Solution/Hint/Comment: From (4) it is obvious that $\iota_v w$ is invariant under coordinate transformations.

6. Consider a set of trajectories of a particle given by $x^i(t, \theta_1, \theta_2, \ldots, \theta_{n-1})$, where t is time and θ's are "initial conditions", so that this set covers D.
   Example: $x^1 = t\cos\theta_1, x^2 = t\sin\theta_1$, D is $\mathbb{R}^2$ without origin.
   a) Show that the velocity vector of the particle $\{\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\}$ defines for us tangent vector field, i.e. that it properly transforms under the change

of coordinates (hence the name "tangent vector field" because velocities are vectors tangent to trajectory).

Solution/Hint/Comment: x and y - two coordinate systems. Considering $x = x(y)$: $\frac{dx^i}{dt} = \frac{\partial x^i}{\partial y^j}\frac{dy^j}{dt}$, which is the same as (4), second line.

b) Compute the velocity field for the above-mentioned example in the coordinate system $\{x^1, x^2\}$ and in the polar coordinates. Draw a plot of this vector field.
Solution/Hint/Comment: Below $x^1 \equiv x$, $x^2 \equiv y$.

In the original coordinates the two components of the vector field is $\{\frac{dx}{dt}, \frac{dy}{dt}\}$, by substituting explicit expressions for $x^i(t, \theta_1)$ and differentiating, we get $\{\cos\theta_1, \sin\theta_1\} = \{\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\}$. In this coordinate frame the basis tangent vectors are denoted by ∂_x and ∂_y. So the vector field is

$$\frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y \,.$$

In the polar coordinates, the components of the vector field are $\{\frac{dr}{dt}, \frac{d\varphi}{dt}\}$. We have to find r, φ explicitly as a functions of t and θ_1. From definition of the polar coordinates, $x = r\cos\varphi$, $y = r\sin\varphi$, it is evident that $r = t$, $\varphi = \theta_1$. Therefore, the components of the vector field is $\{1, 0\}$. Since the basis vectors are denoted as ∂_r and ∂_φ, we get $1 \times \partial_r + 0 \times \partial_\varphi =$

$$\partial_r$$

Note the answer in $\{x, y\}$ fame is easy to obtain by applying chain rule:

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y \,.$$

7. Rewrite $x\partial_y - y\partial_x$ in polar coordinates and complex coordinates.
Solution/Hint/Comment: Answer: ∂_ϕ.

Example of code in *Mathematica*:

```
x pd[y]-y pd[x]/.pd[a_]:>
D[ArcTan[y/x],a]pd[\[Phi]]+D[Sqrt[x^2+y^2],a]pd[r]//Simplify
```

The vector field $x\partial_y - y\partial_x = \partial_\phi$ generates rotations. It is very common, and you have to know this relation by heart.

Although the derivative $\frac{d\arctan(x)}{dx} = \frac{1}{1+x^2}$ is an easy one, it is common to forget this relation. To avoid remembering it, let us discover $x\partial_y - y\partial_x = \partial_\phi$ in a bit different fashion. Generically,

$$x\partial_y - y\partial_x = A\partial_r + B\partial_\phi \,,$$

with A, B being some functions of r and ϕ.

It is clear that $(A\partial_r + B\partial_\phi)r = A$. On the other hand, $(x\partial_y - y\partial_x)r = 0$ (simple computation), therefore $A = 0$. To find B, we compute

$$\partial_\phi = \frac{\partial x}{\partial \phi}\partial_x + \frac{\partial y}{\partial \phi}\partial_y = -y\partial_x + x\partial_y\,. \qquad (5)$$

8.* Rewrite $x\partial_y - y\partial_x$, $x\partial_z - z\partial_x$, $y\partial_z - z\partial_y$ in spherical coordinates. Spherical coordinates are defined by $x = r\,\sin\theta\,\cos\phi$, $y = r\,\sin\theta\sin\phi$, $z = r\,\cos\theta$.

Solution/Hint/Comment: Correspondingly: ∂_ϕ, $-\cos\phi\,\partial_\theta + \cot\theta\,\sin\phi\,\partial_\phi$, $-\sin\phi\,\partial_\theta - \cos\phi\,\cot\theta\,\partial_\phi$. Answer was generated by *Mathematica*.

9.* Metric is a tensor field of rank 2, $g_{ij}(x)$, which defines a scalar product of vectors (at each point x): $\langle v, w\rangle \equiv v^i w^j g_{ij}$. This is another example of vector field, now with $k = 2n$.

Find how metric changes with the change of coordinates
Solution/Hint/Comment:

$$g_{ij}(x) \quad = \quad \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}g_{kl}(y) \qquad (6)$$

10.* If in the Descartes coordinates $g_{ij} = \delta_{ij}$, find g explicitly in polar and complex coordinates for the case of 2-dim space and in spherical coordinates for the case of 3-dim space.
Solution/Hint/Comment: From previous exercise we see $g_{ij} = \sum_k \frac{\partial x^k}{\partial y^i}\frac{\partial x^k}{\partial y^j}$.
For polar coordinates $\{r, \phi\}$: $g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, for complex coordinates $\{z, \bar{z}\}$: $g = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for spherical coordinates $\{r, \theta, \phi\}$: $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$.

Integration

1. Is it correct, in general, to write $\int_\gamma y\,dx = y\,x$?
Solution/Hint/Comment: No, except for the case when $y =$ const along the contour of integration. This question was asked because some of students attempted to find f for which $ydx = df$.

2. Compute integral $\oint_\gamma p\,dq$, where γ is a circle of radius R: $p^2 + q^2 = R^2$. Integration is counterclockwise.
Solution/Hint/Comment: πR^2, by Stockes theorem it is the area of the disk surrounded by the circle. Students might be willing to do it differently, e.g. by introducing polar coordinates.

3. Compute integral $\int_\gamma d(x/y)$ for the following 3 contours. Each contour consists of straight lines connecting the following points:

   a) $\{0,0\},\{1,1\}$,

   b) $\{0,0\},\{1,0\},\{1,1\}$,

   c) $\{0,0\},\{0,1\},\{1,1\}$.

   Solution/Hint/Comment: Although the form is exact, integral should be regularised in the vicinity of zero. a)=0, b) diverges, c)=1.

Legendre transform

Legendre transform of function $f(x)$ on the interval I is a function $g(p)$ defined as

$$g(p) = \sup_{x\in I}(x\,p - f(x))\,. \tag{7}$$

1. It is well-defined operation if $\frac{\partial^2 f(x)}{\partial x^2} \geq 0$. Why?

2. Find the Legendre transform of $\sqrt{1+x^2}$. What is the range of x and p for which the Legendre transform is defined?

   Solution/Hint/Comment: First check when Legendre transform is defined. $\sqrt{1+x^2}'' = \frac{1}{(1+x^2)^{3/2}}$, therefore for any x. Note: on the exam you should not do this verification, unless is explicitly asked to. To find the maximum over x of $x\,p - f(p)$, we solve $(x\,p - f(x))' = 0$ which gives the standard $p = f'$, or explicitly

$$p = \frac{x}{\sqrt{1+x^2}}\,. \tag{8}$$

   Now, solve it with respect to x. Write equation for p^2:

$$p^2 = \frac{x^2}{1+x^2} \qquad \to \qquad x^2 = \frac{p^2}{1-p^2}\,.$$

   Then you have $x = \pm\frac{p}{\sqrt{1-p^2}}$, the sign ambiguity appears because we were taking square root. To fix this ambiguity, we note from (8) that p has the same sign as x. Therefore

$$x = \frac{p}{\sqrt{1-p^2}}\,. \tag{9}$$

   Finally, we substitute the found value of x into $x\,p - f(x)$:

$$g(p) = x\,p - \sqrt{1+x^2}\,, \quad \text{for} \quad x = \frac{p}{\sqrt{1-p^2}}\,. \tag{10}$$

We can of course boldly substitute x, but this will result in nested square root structure and a bit painful simplification. It is much easier to note that $\sqrt{1+x^2} = \frac{x}{p} = \frac{1}{\sqrt{1-p^2}}$. Therefore

$$g(p) = \frac{p}{\sqrt{1-p^2}}\,p - \frac{1}{\sqrt{1-p^2}} = -\sqrt{1-p^2}\,, \tag{11}$$

so the answer $g(p) = -\sqrt{1-p^2}$.

On exact differentials

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1. Find all f such that $df = 2\,x\,y\,dx + (x^2 - y^2)dy$.
   Solution/Hint/Comment: $f = x^2\,y - \frac{1}{3}y^3 + \mathrm{const}$

2. Prove that if ω is exact then $\oint_\gamma \omega = 0$ for any closed contour.

3.* Since for exact differential ω one has $\omega_i = \frac{\partial f}{\partial x^i}$, the necessary condition of exactness is

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}\,. \tag{12}$$

   Prove that if (12) holds than $\oint_\gamma \omega = 0$ for any closed contour γ (note that (12) was not proven to be sufficient condition). Consider for this first γ being a square. Then use the argument that any closed contour is a limit of many squares (at this second step you are not required to be rigorous).

4.* Consider $\omega = \frac{x dy - y dx}{x^2+y^2}$. Is condition (12) satisfied? Compute $\int_\gamma \omega$ for contour being a circle of unit radius $x^2 + y^2 = 1$. Do you get zero? Are you happy with the statements that you proved above?
   Solution/Hint/Comment: In polar coordinates, $\omega = d\phi$, so integral will give 2π. The problem is that ω is singular at 0. Statements above apply only for a contour which can be shrinker to point, and no singularities encountered during shrinking.