Tutorial 6 **(1)**

Part I **Solutions**

MA2342

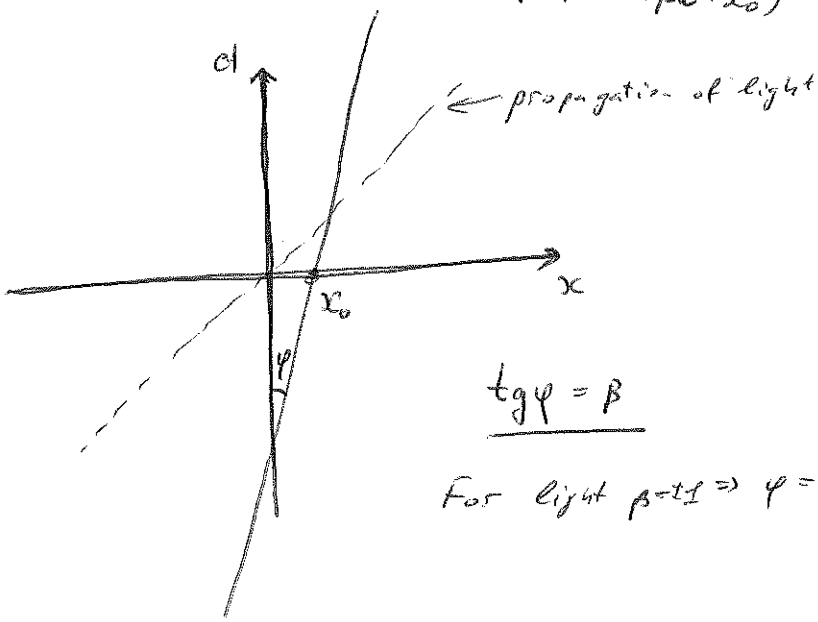
2014

(1) Free particle moving at speed β is ~~g~~ kinematically described as:

$$x = \beta t + x_0$$

This an equation for a line.

Parametric version: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \tau \\ \beta\tau + x_0 \end{pmatrix}$



$\underline{tg\,\varphi = \beta}$

For light $\beta = \pm 1 \Rightarrow \varphi = \pm 45°$

(2) $$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

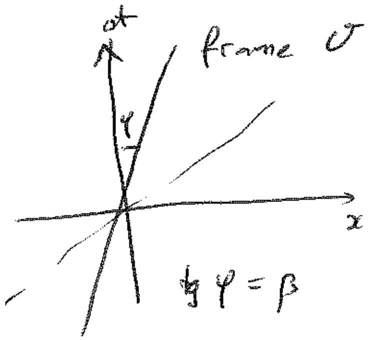
$$(ct')^2 - (x')^2 = (\gamma ct - \beta\gamma x)^2 - ((-\beta\gamma ct)^2 - (\gamma x)^2) =$$

$$= \gamma^2 \left[(ct)^2 - \cancel{2\beta ct x} + (\beta x)^2 - (\beta ct)^2 + \cancel{2\beta ct x} - x^2 \right] =$$

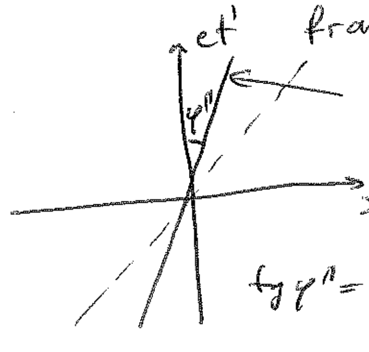
$$= \underbrace{\gamma^2(1-\beta^2)}_{1} \left[(ct)^2 - x^2 \right]$$

$$\left(\text{Indeed: } \gamma^2 = \left(\frac{1}{\sqrt{1-\beta^2}} \right)^2 = \frac{1}{1-\beta^2} \right)$$

(3) (a) Inverse of $\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$ is $\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$



$tg\,\varphi = \beta$



$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \tau \\ \beta''\tau \end{pmatrix}$$

$tg\,\varphi'' = \beta''$

$\boxed{\text{3 - continued}}$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}}_{\substack{\text{Here we should} \\ \text{use } \beta = \beta' \\ \gamma = \gamma(\beta') = \frac{1}{\sqrt{1-\beta^2}}}} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

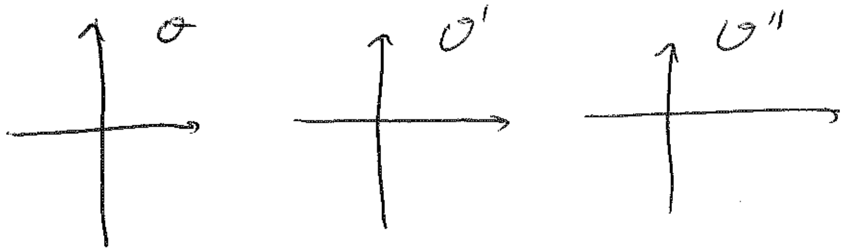
This is because O' moves with respect to O at β'.

$$\begin{pmatrix} \gamma' & \beta'\gamma' \\ \beta'\gamma' & \gamma' \end{pmatrix} \begin{pmatrix} \tau \\ \beta''\tau \end{pmatrix} = \gamma'\tau \begin{pmatrix} 1+\beta''\beta' \\ \beta'+\beta'' \end{pmatrix} = \begin{pmatrix} \tau_{new} \\ \frac{\beta'+\beta''}{1+\beta'\beta''}\,\tau_{new} \end{pmatrix} \Rightarrow \beta$$

$\uparrow$ can be called $\tau_{new} \cdot \frac{1}{1+\beta''\beta'}$

$$\Rightarrow \beta = \frac{\beta'+\beta''}{1+\beta'\beta''}$$ $\leftarrow$ speed of a particle in ref frame O

(b)



β' : speed of O' w.r.t O (with respect to)

β'' : speed of O'' w.r.t. O'

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma' & \beta'\gamma' \\ \beta'\gamma' & \gamma' \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma'' & \beta''\gamma'' \\ \beta''\gamma'' & \gamma'' \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \end{pmatrix}$$

$O' \to O \qquad\qquad O'' \to O'$

Hence: $O'' \to O$:
$$\begin{pmatrix} ct \\ x \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma' & \beta'\gamma' \\ \beta'\gamma' & \gamma' \end{pmatrix} \begin{pmatrix} \gamma'' & \beta''\gamma'' \\ \gamma''\beta'' & \gamma'' \end{pmatrix}}_{(*)} \begin{pmatrix} ct'' \\ x'' \end{pmatrix}$$

3b - continued

Tutorial 6 (3)

Part I Solutions

MA 2342

$(*)$: $\begin{pmatrix} \gamma' & \beta'\gamma' \\ \beta'\gamma' & \gamma' \end{pmatrix}\begin{pmatrix} \gamma'' & \beta''\gamma'' \\ \beta''\gamma'' & \gamma'' \end{pmatrix} =$

$$= \gamma'\gamma''\begin{pmatrix} 1 & \beta' \\ \beta' & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta'' \\ \beta'' & 1 \end{pmatrix} = \gamma'\gamma''\begin{pmatrix} 1+\beta'\beta'' & \beta'+\beta'' \\ \beta'+\beta'' & 1+\beta'\beta'' \end{pmatrix} =$$

$$= (\gamma'\gamma'')\cdot(1+\beta'\beta'')\begin{pmatrix} 1 & \frac{\beta'+\beta''}{1+\beta'\beta''} \\ \frac{\beta'+\beta''}{1+\beta'\beta''} & 1 \end{pmatrix}$$

$$(1+\beta'\beta'')\gamma'\gamma'' = \frac{1}{\sqrt{1-(\beta')^2}}\,\frac{1+\beta'\beta''}{\sqrt{1-(\beta'')^2}} = \frac{\sqrt{(1+\beta'\beta'')^2}}{\sqrt{1-(\beta')^2-(\beta'')^2+(\beta'\beta'')^2}} =$$

$$= \frac{\sqrt{(1+\beta'\beta'')^2}}{\sqrt{(1+\beta'\beta'')^2-(\beta'+\beta'')^2}} = \frac{1}{\sqrt{1-\left(\frac{\beta'+\beta''}{1+\beta'\beta''}\right)^2}}$$

$$(*) = \frac{1}{\sqrt{1-\left(\frac{\beta'+\beta''}{1+\beta'\beta''}\right)^2}}\begin{pmatrix} 1 & \frac{\beta'+\beta''}{1+\beta''\beta'} \\ \frac{\beta''+\beta'}{1+\beta'\beta''} & 1 \end{pmatrix}$$ as expected.

4 See HW 7.

5 See HW 7.

6 see HW 7. $(A=m)$

7 We can check: $\gamma^2-\beta^2\gamma^2=1$. Trigonometric relation is $\cosh^2\theta-\sinh^2\theta=1$. $\longrightarrow$ this what enables and suggest the transformation.

$\theta \to \theta'$:

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \cosh\theta_\beta & \sinh\theta_\beta \\ \sinh\theta_\beta & \cosh\theta_\beta \end{pmatrix}\underbrace{\begin{pmatrix} E/c \\ p \end{pmatrix}}_{\begin{pmatrix} mc\cosh\theta \\ mc\sinh\theta \end{pmatrix}}$$

$\boxed{7\text{ - continued}}$

$$\begin{pmatrix} \cosh\theta_\beta & \sinh\theta_\beta \\ \sinh\theta_\beta & \cosh\theta_\beta \end{pmatrix} \begin{pmatrix} mc\cosh\theta \\ mc\sinh\theta \end{pmatrix} =$$

$$= mc \begin{pmatrix} \cosh\theta_\beta \cosh\theta + \sinh\theta_\beta \sinh\theta \\ \sinh\theta_\beta \cosh\theta + \sinh\theta \cosh\theta_\beta \end{pmatrix} = mc \begin{pmatrix} \cosh(\theta+\theta_\beta) \\ \sinh(\theta+\theta_\beta) \end{pmatrix}$$

$\theta \to \theta + \theta_\beta$ (that's how rapidity changes).