1 Substitute ansatz $\Psi = e^{i(\vec{k}\vec{x} - \omega t)}$

into equation:

$$\frac{\partial}{\partial t}\Psi = -i\omega\Psi \; ; \quad \frac{\partial^2}{\partial t^2}\Psi = (-i\omega)^2\Psi = -\omega^2\Psi$$

$$\frac{\partial^2}{\partial x_i^2}\Psi = -k_i^2\Psi$$

Then equation $\displaystyle\frac{\partial^2\Psi}{\partial t^2} - \sum_{i=1}^{3} A_i \frac{\partial^2\Psi}{\partial x_i^2} = 0$

becomes

$$\left(-\omega^2 + \sum_{i=1}^{3} A_i k_i^2\right)\Psi = 0$$

Hence, dispersion relation is

$$\boxed{\omega = \pm\sqrt{\sum_{i=1}^{3} A_i k_i^2}}$$

~~("+" sometimes called positive energy~~

"+" - wave propagates one direction

"−" - wave propagates opposite direction.

Phase speed is $\displaystyle\vec{v}_p = \frac{\omega}{|\vec{k}|}\vec{n}_k = \pm\frac{\sqrt{\sum_{i=1}^{3} A_i k_i^2}}{|\vec{k}|}\vec{n}_k$

Group speed is $\displaystyle\vec{v}_g = \frac{\partial\omega}{\partial\vec{k}} = \pm\frac{\partial}{\partial\vec{k}}\sqrt{\dots} \; ; \quad (v_g)_i = \pm\frac{A_i k_i}{\sqrt{\sum_i A_i k_i^2}}$

$\cancel{\dfrac{\partial\omega}{\partial\vec{k}}}$

Note $\vec{v}_g \neq \vec{v}_p$ (for generic A)

2 (a)

(*) $(\nabla W)^2 = 2m(E - V) \quad ; \quad V = \frac{\alpha}{r^2}$

$$\vec{\nabla} W = \frac{\partial W(r)}{\partial \vec{x}_3} = \frac{dW}{dr}\frac{\partial r}{\partial \vec{x}} = \frac{dW}{dr}\frac{\partial}{\partial \vec{x}}\sqrt{\vec{x}^2} = \frac{\vec{x}}{r}\frac{dW}{dr}$$

using that W depends only on r

$$(\vec{\nabla} W)^2 = \left(\frac{\vec{x}}{r}\right)^2 \left(\frac{dW}{dr}\right)^2 = \left(\frac{dW}{dr}\right)^2$$

Hence (*) becomes:

$$\left(\frac{dW}{dr}\right)^2 = 2m\left(E - \frac{\alpha}{r^2}\right)$$

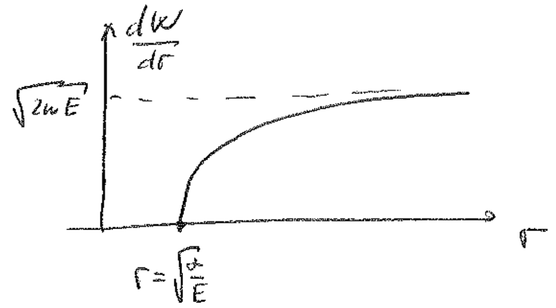
$$dW = \sqrt{2m\left(E - \frac{\alpha}{r^2}\right)}\, dr$$

$$\int dW = \int \sqrt{\ldots}\, dr$$

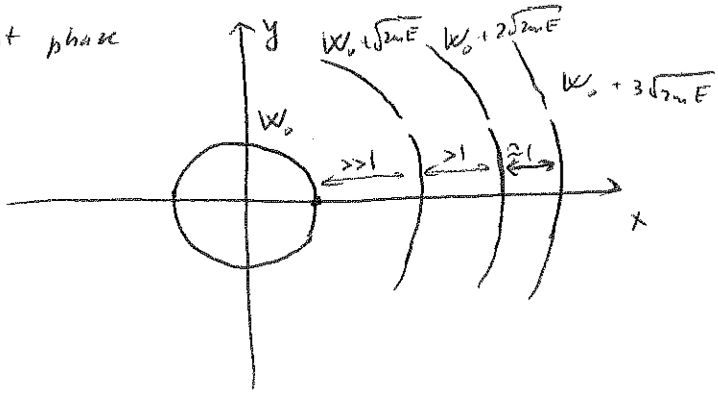
$$W = \sqrt{2m}\left(r\sqrt{E - \frac{\alpha}{r^2}} + \sqrt{\alpha}\,\mathrm{ArcTan}\left(\frac{\sqrt{\alpha}}{r\sqrt{E - \frac{\alpha}{r^2}}}\right)\right) + \text{const.}$$

(b) These surfaces should be circles because $W = W(r)$.

What is interesting, is distance between them



Constant phase

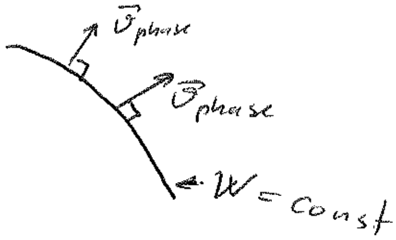


2 (c)

To compute (make sense) phase and group velocities, one should work in the linear approximation:

$$S = W(x) - Et = W(x_0) + \underbrace{(\vec{\nabla} W)}_{\vec{k}}(\vec{x} - \vec{x}_0) - \underbrace{E}_{\omega} t + O((\vec{x}-\vec{x}_0)^2)$$

$$\vartheta_{phase} = \frac{\omega}{|k|} n_k = \frac{E}{|\vec{\nabla} W|} \vec{n}_k = \frac{E}{(\vec{\nabla} W)^2} \vec{\nabla} W = \frac{\cancel{E}}{2m(E-V)}$$



$$|\vartheta_{phase}| = \frac{E}{|\vec{\nabla} W|} = \frac{E}{\sqrt{2m(E-V)}}$$

$$\vec{\vartheta}_{group} = \frac{\partial \omega}{\partial \vec{k}} = \frac{\partial E}{\partial \vec{\nabla} W} = \frac{\partial\left(V + \frac{1}{2m}(\vec{\nabla} W)^2\right)}{\partial \vec{\nabla} W} = \frac{1}{m} \vec{\nabla} W$$

(used: $E = V + \frac{1}{2m}(\vec{\nabla} W)^2$)

Recall that $\vec{\nabla} W = \vec{p}$ (from interpretation of S as generator of canonical transformation)

Therefore $\vec{\vartheta}_{group} = \frac{\vec{p}}{m}$ — it is nothing but a speed of a particle!

Tutorial 5

solutions

3 $$-i\hbar \frac{\partial \Psi}{\partial t} - \frac{\hbar^2}{2m}\Delta\Psi + V(x)\,\Psi = 0 \qquad (*)$$

As requested, plugging explicitly

$$\Psi = A\, e^{i/\hbar\, S}$$

$$\frac{\partial \Psi}{\partial t} = \frac{\partial A}{\partial t}e^{u} + \frac{i}{\hbar}\frac{\partial S}{\partial t}A e^{u} = \left(\frac{1}{A}\frac{\partial A}{\partial t} + \frac{i}{\hbar}\frac{\partial S}{\partial t}\right)\Psi$$

$$\vec\nabla\Psi = (\text{the same procedure}) = \left(\frac{1}{A}\vec\nabla A + \frac{i}{\hbar}\vec\nabla S\right)\Psi$$

$$\Delta\Psi = \vec\nabla(\vec\nabla\Psi) = \left[\vec\nabla\left(\frac{1}{A}\vec\nabla A + \frac{i}{\hbar}\vec\nabla S\right)\right]\Psi + \left(\frac{1}{A}\vec\nabla A + \frac{i}{\hbar}\vec\nabla S\right)\vec\nabla\Psi =$$

$$= \left[\Delta \log A + \frac{i}{\hbar}\Delta S\right]\Psi + \left((\vec\nabla \log A)^2 - \frac{1}{\hbar^2}(\vec\nabla S)^2 + \frac{2i}{\hbar}(\vec\nabla S)(\vec\nabla \log A)\right)\Psi$$

$$\Delta \log A + (\vec\nabla \log A)^2 = \vec\nabla\left(\frac{1}{A}\vec\nabla A\right) + \left(\frac{1}{A}\vec\nabla A\right)^2 =$$

$$= \cancel{-\frac{1}{A^2}(\vec\nabla A)^2} + \frac{1}{A}\Delta A + \cancel{\left(\frac{1}{A}\vec\nabla A\right)^2} = \frac{1}{A}\Delta A.$$

$$\Delta\Psi = \left[\frac{1}{A}\Delta A + \frac{i}{\hbar}\left(\Delta S + 2(\vec\nabla S)(\vec\nabla \log A)\right) - \frac{1}{\hbar^2}(\vec\nabla S)^2\right]\Psi$$

Making correct powers of $\hbar$!

$$-i\hbar\frac{\partial\Psi}{\partial t} = \left(\hbar^1\left(-i\frac{\partial \log A}{\partial t}\right) + \hbar^0\frac{\partial S}{\partial t}\right)\Psi$$

$$-\frac{\hbar^2}{2m}\Delta\Psi = \left[\hbar^2\left(-\frac{1}{2m}\frac{\Delta A}{A}\right) + \hbar\left[-\frac{i}{2m}\left(\Delta S + 2(\vec\nabla S)(\vec\nabla \log A)\right)\right] + \hbar^0\cdot\frac{(\vec\nabla S)^2}{2m}\right]\Psi$$

$$+V(x)\Psi = +V(x)\Psi$$

Collecting together and rewriting (*)

$$(*):\quad 0 = \hbar^0\left(\frac{\partial S}{\partial t} + \frac{(\vec\nabla S)^2}{2m} + V(x)\right) + (-i\hbar)\left[\frac{1}{A}\left(\frac{\partial A}{\partial t} + \frac{2(\vec\nabla S)(\vec\nabla A)}{2m} + \frac{A\cdot\Delta S}{2m}\right)\right] +$$

$$+\ \hbar^2\left(-\frac{1}{2m}\frac{\Delta A}{A}\right)$$

3 - continued

Leading order: $\hbar^0$

$$\frac{\partial S}{\partial t} + \frac{(\vec{\nabla} S)^2}{2m} + V(x) = 0$$

This is just a Hamilton-Jacobi equation!

So the phase of Quantum-Mechanical wave in the quasiclassical ($\hbar \to 0$) approximation
$\equiv$ eikonal
coincides with action of classical mechanics.

Subleading order:

$$\hbar^1 \quad \frac{\partial A}{\partial t} + \frac{2(\vec{\nabla} S)(\vec{\nabla} A)}{2m} + \frac{A \, \Delta S}{2m} = 0$$

As suggested, change variable:

$$A = \sqrt{\rho}$$

$$\frac{\partial A}{\partial t} = \frac{1}{2\sqrt{\rho}} \frac{\partial \rho}{\partial t} \; ; \; \vec{\nabla} A = \frac{1}{2\sqrt{\rho}} \vec{\nabla} \rho \; ; \; A = \frac{1}{\sqrt{\rho}} \cdot \sqrt{\rho}$$

we get:

$$\frac{\partial \rho}{\partial t} + \left(\frac{\vec{\nabla} S}{m}\right)(\vec{\nabla} \rho) + \rho \cdot \vec{\nabla} \cdot \left(\frac{\vec{\nabla} S}{m}\right) = 0$$

$$\Updownarrow$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\left(\rho \frac{\vec{\nabla} S}{m}\right)$$

But $\frac{1}{m} \vec{\nabla} S = \vec{v}_g$ - velocity of the "particle"!
[group velocity]

$$\boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0}$$ - continuity equation!

So ρ is the "particle density" (in some sense). Quantum mechanics interpretation is that this is a <u>probability density</u> (to find a particle at given position).

3 - continued

(can skip)

2nd subleading order:

writing ~~eq~~ $-\frac{1}{2m}\frac{\Delta A}{A} = 0$ is not correct.

The reason is that S receives quantum corrections:

$$S = S_{classical} + \hbar^2 \delta S + \ldots$$

Correct equation (up to 2nd subleading order) is

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m}\frac{\Delta A}{A} = 0 \qquad (*)$$

with $S = S + \hbar^2 \delta S$ (i.e 2 equations: HJ for S and (*) for δS).

A should be found from the continuity equation

$-\frac{\hbar^2}{2m}\frac{\Delta A}{A}$ is sometimes called quantum potential.