Tutorial 4 (1)

Solutions

1. $\{Q, P\}_{q,p} = \{q, p\}_{q,p} = 1 \Rightarrow$ canonical.

[this is a check that transformation preserves symplectic structure]

$dS_1 = p\,dq - P\,dQ$ = (in this case) = $p\,dq - p\,dq = 0 \Rightarrow S_1$ is ill-defined

time-independent case

$dS_2 = p\,dq + Q\,dP = \underbrace{P\,dq + q\,dP}_{} = d(qP) \Rightarrow S_2 = qP$

we make this choice because S_2 is a function of q, P explicitly, and S_3 is a function of p, Q explicitly

$dS_3 = -q\,dp - P\,dQ = \overbrace{-Q\,dp - p\,dQ} = d(-pQ) \Rightarrow S_3 = -pQ$

$dS_4 = -q\,dp + Q\,dP = 0 \Rightarrow S_4$ is ill-defined

Comment: $S_1(q, Q)$ is well-defined only in the case when q and Q can be considered as independent variables.

2. $\{Q, P\}_{q,p} = \{-\alpha p, \alpha q\}_{q,p} = -\alpha^2 \{p, q\}_{q,p} = \alpha^2 \overset{\text{should be} \Leftrightarrow \text{transformation is canonical}}{=} 1 \Rightarrow \alpha = \pm 1$

$\{f, g\}_{q,p} \equiv \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$. I write subscript $_{q,p}$ explicitly because in principle there is also $\{\cdot,\cdot\}_{Q,P}$

fast (in this case) approach

$dS_4 = -q\,dp + Q\,dP = -\frac{1}{\alpha} P\,dp - \alpha p\,dP$ = (using $\alpha = \pm 1$) =

$= -\alpha (P\,dp + p\,dP) = -\alpha\, d(pP) \Rightarrow S_4 = -\alpha p P$

Systematic approach:

$\left.\frac{\partial S_4}{\partial p}\right|_P = -q = -\frac{1}{\alpha} P = (\alpha = \pm 1) = -\alpha P \Rightarrow$

$\Rightarrow S_4 = \int -\alpha P\,dp + h(P) =$

$= -\alpha P p + h(P)$

Origin of function $h(P)$:

if you solve $\frac{dy(x)}{dx} = f(x)$, you write $y = \int f(x)dx + \text{const}$

In our case it is $\left.\frac{\partial y(x,z)}{\partial x}\right|_{\text{at const } z} = f(x) \Rightarrow y = \int f(x)dx + \underbrace{\text{const}(z)}_{\text{function of } z \text{ in general}}$

2-continued: Fix $h(P)$

From $S_4(p, P) = -\alpha P p + h(P)$: $\left.\frac{\partial S_4}{\partial P}\right|_p = -\alpha p + \frac{dh}{dP}$

From $dS_4 = -q\,dp + Q\,dP$: $\left.\frac{\partial S_4}{\partial P}\right|_p = Q = -\alpha p$ $\Rightarrow$

($Q = -\alpha p$ = condition of the problem)

$\Rightarrow \frac{dh}{dP} = 0 \Rightarrow h = \text{const}$. Can choose $h = 0$ because any generating function is defined up to a constant, physical interest only dS has.

3 Checking of being canonical from exactness of diff. form condition:

$$p\,dq - P\,dQ = p\,dq - (\cos\theta\, p + \sin\theta\, q)\, d(-\sin\theta\, p + \cos\theta\, q) =$$

[to check exactness, we can choose any pair of variables. So we choose $\{p, q\}$, because it is the easiest way. It is however not a way to find S_1 (at least, it is not enough)]

$$= p\,dq + \sin\theta\cos\theta \underbrace{p\,dp}_{\substack{= \frac{1}{2}dp^2 \\ \text{Exact}}} - \sin\theta\cos\theta \underbrace{q\,dq}_{\substack{= \frac{1}{2}dq^2 \\ \text{Exact}}} - \cos^2\theta\, p\,dq + \sin^2\theta\, q\,dp =$$

$$= (1-\cos^2)\, p\,dq + \sin^2\theta\, q\,dp + \text{"Exact"} = \sin^2\theta \underbrace{(p\,dq + q\,dp)}_{\text{Exact}} + \text{"Exact"} \Leftarrow \text{exact}.$$

3-continued

Conclusion: this transformation is canonical for any θ.

To find S_1, we use $\left.\frac{\partial S_1}{\partial q}\right|_Q = \theta p$ (a); $\left.\frac{\partial S_1}{\partial Q}\right|_q = -P$ (b)

Hence we need to express p an P as functions of q and Q.

~~$P = \cos\theta p - q$~~ $p = \frac{\cos\theta q - Q}{\sin\theta} = \cot\theta\, q - \frac{1}{\sin\theta} Q$

$$P = \cos\theta\, p + \sin\theta\, q = \cos\theta\left(\frac{\cos\theta q - Q}{\sin\theta}\right) + \sin\theta\, q =$$

$$= \frac{1}{\sin\theta} q - \cot\theta\, Q$$

(a): $\left.\frac{\partial S_1}{\partial q}\right|_Q = \cot\theta\, q - \frac{1}{\sin\theta} Q \Rightarrow \underline{S_1} = \int \left(\cot\theta\, q - \frac{Q}{\sin\theta}\right) dq + h(Q) =$

$$= \frac{1}{2}\cot\theta\, q^2 - \frac{1}{\sin\theta} Q q + h(Q) \quad (*)$$

from (*): $\left.\frac{\partial S_1}{\partial Q}\right|_q = -\frac{1}{\sin\theta} q + \frac{dh}{dQ}$

from (b): $\left.\frac{\partial S_1}{\partial Q}\right|_q = \cot\theta\, Q - \frac{1}{\sin\theta} q \Rightarrow$

$\Rightarrow h = \frac{1}{2}\cot\theta\, Q^2$

Answer: $\boxed{S_1 = \frac{1}{2}\cot\theta(q^2 + Q^2) - \frac{1}{\sin\theta} Q q}$

4 - discussed in solution to HW5

5. (a) We have to understand how $\{Q,P\}$ differs from 1 and suggest a correction.

$$\{Q,P\} = \{q+p\ ,\ 3q\,(e^{(q+p)^5}+1) + p\,(3e^{(q+p)^5}+1)\} =$$

$$= 3(\cancel{3}e^{(q+p)^5}+1)\{q+p,q\} + (3e^{(q+p)^5}+1)\{q+p,p\} =$$

this goes out because

$$\{q+p,q+p\}=0 \Rightarrow \{q+p, f(q+p)\}=0 \Rightarrow \{q+p, h\cdot f(q+p)\} = f(q+p)\{q+p,h\}$$

$$= 3(\cancel{3}e^{(q+p)^5}+1)\cdot(-1) + (3e^{(q+p)^5}+1)\cdot(1) \ = (\text{want}) = 1\ ?$$

$$\|(\text{really}) = -2$$

So, a possible correction is:

$$\{3q(e^{(q+p)^5}+1),\ (3e^{(q+p)^5}+4)\}$$

here.

[of course, it is not the only option]

(b) $$\{Q,P\} = \{\log p^{2014} q^{2013}\ ,\ qp\} =$$

$$= \{2013\log(qp) + \log p\ ,\ qp\} = \{\log p, qp\} =$$

$$= p\{\log p, q\} = p\cdot\frac{1}{p}\{p,q\} = -1$$

We want $+1$, the possible change: $\boxed{\log q^{2014}\ p^{2013}}$

6 Equations of motion:

$$\dot{q} = \frac{p}{m} \; ; \; \dot{p} = 0$$

$$\Downarrow$$

should be

$$q(t) = \frac{p}{m} t + c \; ; \; q(t_i) = q_i = \frac{p}{m} t_i + c$$

$$p = \text{const} \qquad q(t_f) = q_f = \frac{p}{m} t_f + c \Bigg\} \Rightarrow$$

$$\Rightarrow \frac{q_f - q_i}{t_f - t_i} = \frac{p}{m} \; ; \; \boxed{p = m \frac{q_f - q_i}{t_f - t_i}}$$

$$\frac{m q_i}{t_i} - \frac{m q_f}{t_f} = c\left(\frac{m}{t_i} - \frac{m}{t_f}\right)$$

$$\boxed{c = \frac{t_f q_i - t_i q_f}{t_f - t_i}}$$

check:

$$q(t) = \frac{p}{m} t + c = \frac{q_f - q_i}{t_f - t_i} t + \frac{t_f q_i - t_i q_f}{t_f - t_i}$$

$$q(t_f) = \frac{q_f - q_i}{t_f - t_i} t_f + \frac{t_f q_i - t_i q_f}{t_f - t_i} = q_f \checkmark$$

$$q(t_i) = \frac{q_f - q_i}{t_f - t_i} t_i + \frac{t_f q_i - t_i q_f}{t_f - t_i} = q_i \checkmark$$

$$\mathcal{L} = p\dot{q} - \frac{p^2}{2m} = m \frac{q_f - q_i}{t_f - t_i} \cdot \frac{q_f - q_i}{t_f - t_i} - \frac{1}{2m}\left(m \frac{q_f - q_i}{t_f - t_i}\right)^2 = \frac{m}{2}\left(\frac{q_f - q_i}{t_f - t_i}\right)^2$$

$$\underline{S} = \int_{t_i}^{t_f} \mathcal{L} dt = \underline{\frac{m}{2} \frac{(q_f - q_i)^2}{t_f - t_i}} \leftarrow \text{required answer}$$

Now replacing: $q_f \to q$, $q_i \to Q$, $t_f - t_i \to t$

$$\boxed{S(q, Q, t) = \frac{m}{2} \frac{(q - Q)^2}{t}}$$

6 - continued

$$p = \frac{\partial S}{\partial q} = \frac{m}{t}(q - Q) \qquad (*)$$

$$P = -\frac{\partial S}{\partial Q} = \frac{m}{t}(q - Q)$$

Q, P as functions of q, p, t: $\quad Q = q - \frac{p}{m} t$

$$P = p$$

q, p as functions of Q, P, t: $\quad \boxed{\begin{aligned} q &= Q + \frac{P}{m} t \\ p &= P \end{aligned}}$

Conclusion: this canonical transformation just solves equation of motion! (reminder: $Q = q_i$, $P = p_i$)

$$\frac{\partial S}{\partial t} = \mathcal{H}' - \mathcal{H} \Rightarrow \mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t} = \frac{p^2}{2m} - \frac{m}{2}\frac{(Q-q)^2}{t^2} =$$

$$= (\text{using } *) = \frac{p^2}{2m} - \frac{m}{2}\left(\frac{p}{m}\right)^2 = 0.$$

in general, you have now express the obtained expression in Q, P, to get $\mathcal{H}'(Q, P)$

$\mathcal{H}' = 0$ (Ideed, meaning of Q and P are initial conditions. They do not evolve in time).

Conclusion: if you are able to find S, you solve your system.

S is usually found by solving HJ equations.

Here we just check that HJ are satisfied:

$$\frac{\partial S}{\partial t} + \mathcal{H}\left(q, \frac{\partial S}{\partial q}\right) = 0 \quad \leftarrow \text{should be}$$

Explicitly for $\mathcal{H}(q,p) = \frac{p^2}{2m}$:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 = \left(\text{substituting explicit expression for } S\right) =$$

$$= -\frac{m}{2}\frac{(q-Q)^2}{t^2} + \frac{1}{2m}\left(\frac{m}{2}(q-Q)\right)^2 = 0 \checkmark$$

7* (can skip)

EOM for Harmonic oscillator: $\dot{q} = p$

$$\dot{p} = -\omega^2 q$$

You should know that solution can be written in the form $q = A\cos(\omega t + \varphi)$ [For instance, you solved $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$, harmonic oscillator is of the same type]

We will just check that it is true:

$$p = \dot{q} = -\omega A \sin(\omega t + \varphi)$$

$$\dot{p} = -\omega^2 A \cos(\omega t + \varphi) = q \cdot (-\omega^2) \checkmark$$

We have to express A and φ through initial conditions.

$$\left.\begin{aligned} q(t_i) &= q_i = A\cos(\omega t_i + \varphi) \\ q(t_f) &= q_f = A\cos(\omega t_f + \varphi)\end{aligned}\right\} (*)$$

It is not very pleasant to solve $(*)$ for A and t_i, and think that we will hence keep $(*)$ as it is and it is a parametric definition of A, φ.

$$\mathcal{L} = p\dot{q} - \mathcal{H} = (-\omega A \sin(\omega t + \varphi))(-\omega A \sin(\omega t + \varphi)) - \frac{1}{2}\Big[\overbrace{\omega^2 A^2 \sin^2}^{p^2} + \overbrace{\omega^2 A^2 \cos^2}^{q^2}\Big] =$$

$$= +\omega^2 A^2 \sin^2(\omega t + \varphi) - \frac{1}{2}\omega^2 A^2$$

7 - continued

Use that $\sin^2\alpha = \frac{1}{2}(1-\cos(2\alpha))$

$$S = \int_{t_i}^{t_f} (p\dot{q} - \mathcal{H})dt = \int_{t_i}^{t_f} \frac{\omega^2 A^2}{2}\left(1-\cos(2\omega t + 2\varphi)\right) dt - \frac{1}{2}\omega^2 A^2 (t_f - t_i) =$$

$$= - \int_{t_i}^{t_f} \frac{\omega^2 A^2}{2} \cos(2\omega t + 2\varphi)\, dt = - \frac{\omega A^2}{4}\left(\sin(2\omega t_f + 2\varphi) - \sin(2\omega t_i + 2\varphi)\right)$$

~~Since this does not depend on time explicitly~~

Apparent Dependence separately on t_f and t_i is misleading. The final answer is expressed in terms of A, t_i, t_f, φ. If we express it in terms of q_f, q_i, t_f, t_i, then we will see that it depends only ~~t~~ on $t = t_f - t_i$

Indeed, for A one has:

$$\arccos\left(\frac{A}{q_f}\right)^{-1} - \arccos\left(\frac{A}{q_i}\right)^{-1} = \omega(t_f - t_i) = \omega t$$

So $A = A(q_f, q_i, t)$.

Answer actually depends on combination $\omega t_i + \varphi$ and $\omega t_f + \varphi$, but these combinations are functions, respectively, of q_i/A and q_f/A, see (*). For A we proved above that $A = A(q_f, q_i, t)$.

~~Because of this~~ After this observation, we will put $t_i = 0$, so our parameterisation becomes:

$$\left.\begin{aligned} Q &= q_i = A\cos\varphi \\ q &= q_f = A\cos(\omega t + \varphi) \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} &\arccos\left(\frac{A}{q_f}\right)^{-1} - \arccos\left(\frac{A}{q_i}\right)^{-1} = \omega t \\ &\varphi = \arccos\frac{q_i}{A} \end{aligned}\right.$$

$$S = -\frac{\omega A^2}{4}\left(\sin(2\omega t + 2\varphi) - \sin(2\varphi)\right)$$

7 - continued

From the found value of S we should read off p & P, and $\mathcal{H}-\mathcal{H}'$.

Let's do it in the following way: ~~Consider t fixed~~ (to ignore dt term below)

~~$dS = pdq - PdQ$ ← want this~~ form

~~On the other hand:~~

~~$dS =$~~

$$dS = pdq - PdQ - (\mathcal{H}-\mathcal{H}')dt$$ ← want this form

But we can produce:

$$dS = -\frac{\omega A}{2}\Big(\underbrace{\sin(2\omega t+2\varphi) - \sin(2\varphi)}_{2\sin(\omega t)\cos(\omega t+2\varphi)}\Big)dA - \frac{\omega A^2}{2}\Big(\overbrace{\cos(2\omega t+2\varphi) - \cos(2\varphi)}^{-2\sin(\omega t)\sin(\omega t+2\varphi)}\Big)d\varphi - \frac{\omega^2 A^2}{2}\cos(2\omega t+2\varphi)\,dt$$

Express dA in terms of dq, dQ, dt is a bit complicated. So we do inverse:

$$dq = \cos(\omega t+\varphi)\,dA - \omega A\sin(\omega t+\varphi)\,dt - A\sin(\omega t+\varphi)\,d\varphi$$

$$dQ = \cos(\varphi)\,dA - A\sin(\varphi)\,d\varphi$$

Conclusion:

$$p\cdot\cos(\omega t+\varphi) - P\cdot\cos(\varphi) = -\omega A\sin(\omega t)\cos(\omega t+2\varphi)$$

$$p\cdot(-A\sin(\omega t+\varphi)) - P\cdot(-A\sin\varphi) = \omega \cancel{A^2}\sin(\omega t)\sin(\omega t+2\varphi)$$
$-A$

$$P = \frac{-\omega A\sin(\omega t)\left[+\frac{\cos(\omega t+2\varphi)}{\cos(\omega t+\varphi)} - \frac{\sin(\omega t+2\varphi)}{\sin(\omega t+\varphi)}\right]}{-\left[\frac{\cos(\varphi)}{\cos(\omega t+\varphi)} - \frac{\sin\varphi}{\sin(\omega t+\varphi)}\right]} = -\omega A\sin(\omega t)\cdot\frac{\sin\varphi}{\sin\omega t} = -\omega A\sin(\varphi)$$

$$p = -\omega A\cdot\sin(\omega t)\cdot\frac{\frac{\cos(\omega t+2\varphi)}{\cos(\varphi)} - \frac{\sin(\omega t+2\varphi)}{\sin\varphi}}{\frac{\cos(\omega t+\varphi)}{\cos\varphi} - \frac{\sin(\omega t+\varphi)}{\sin(\varphi)}} = -\omega A\cdot\sin(\omega t+\varphi)$$

Conclusion:
$$\boxed{\begin{aligned} P &= -\omega A \sin\varphi \\ p &= -\omega A \sin(\omega t+\varphi) \end{aligned}}$$
(as it should)

Now we compare dt terms:

$$p\cdot\left[-\omega A \sin(\omega t+\varphi)\right] - (\mathcal{H}-\mathcal{H}') = -\frac{\omega^2 A^2}{2}\cos(2\omega t+2\varphi)$$

$$\mathcal{H}' = \mathcal{H} + (\omega A)^2\left[-\frac{1}{2}\cos(2\omega t+2\varphi) - \sin^2(\omega t+\varphi)\right] =$$

$$= \mathcal{H} - \frac{(\omega A)^2}{2}$$

But $\mathcal{H} = \frac{1}{2}(p^2+\omega^2 q^2) = \frac{1}{2}(\omega A)^2$

Hence $\boxed{\mathcal{H}'=0}$ (as it should).

Checking HJ: $\quad \frac{\partial S}{\partial t} + \mathcal{H}\left(q, \frac{\partial S}{\partial q}\right) = 0 \quad$ for $\quad \mathcal{H} = \frac{1}{2}(\omega^2 q^2 + p^2)$

$$\frac{\partial S}{\partial t} + \frac{\omega^2}{2}q^2 + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 = \underbrace{\frac{\partial S}{\partial t}}_{= -\mathcal{H}} + \overbrace{\frac{1}{2}(\omega^2 q^2 + p^2)}^{2\mathcal{H}} = 0$$

(because $\mathcal{H}'=0$, already checked)

already computed explicitly

In our implicit parameterisatio HJ is automatic.

8.9 What we actually want, is to show that

$$\begin{aligned}\{Q_i, P_j\}_{p,q} &= \delta_{ij}\\ \{Q_i, Q_j\}_{p,q} &= 0\\ \{P_i, P_j\}_{p,q} &= 0\end{aligned} \iff \alpha = p\,dq - P\,dQ \text{ is exact differential.}$$

(*) (**)

First, (*) is equivalent to $\omega_{\alpha\beta} \dfrac{\partial X^\alpha}{\partial x^{\alpha'}} \dfrac{\partial X^\beta}{\partial x^{\beta'}} = \omega_{\alpha'\beta'}$ (***),

where $X = (Q, P)$ (2n-dim vector) [see lecture notes online]
$x = (q, p)$

Now, rewrite $dQ = \dfrac{\partial Q}{\partial q_r} dq_r + \dfrac{\partial Q}{\partial p_r} dp_r$ ($Q(\vec{q}, \vec{p})$ is thought as function of q and p in our derivation)

$$\alpha = p\,dq - P\,dQ = \left(p_r - P_i \frac{\partial Q_i}{\partial q_r}\right) dq_r - P_i \frac{\partial Q_i}{\partial p_r} dp_r$$

Conditions of exactness:

$$\frac{\partial}{\partial q_s}\left(p_r - P_i \frac{\partial Q_i}{\partial q_r}\right) = \frac{\partial}{\partial q_r}\left(p_s - P_i \frac{\partial Q_i}{\partial q_s}\right) \quad (a)$$

$$\frac{\partial}{\partial p_s}\left(- P_i \frac{\partial Q_i}{\partial p_r}\right) = \frac{\partial}{\partial p_r}\left(- P_i \frac{\partial Q_i}{\partial p_s}\right) \quad (b)$$

$$\frac{\partial}{\partial p_s}\left(p_r - P_i \frac{\partial Q_i}{\partial q_r}\right) = \frac{\partial}{\partial q_r}\left(- P_i \frac{\partial Q_i}{\partial p_s}\right) \quad (c)$$

Consider for instance (c). The rest is done by analogy:

$$(c)\quad \underbrace{\frac{\partial p_r}{\partial p_s}}_{=\delta_{sr}} = \frac{\partial}{\partial p_s}\left(P_i \frac{\partial Q_i}{\partial q_r}\right) - \frac{\partial}{\partial q_r}\left(P_i \frac{\partial Q_i}{\partial p_s}\right) = \frac{\partial P_i}{\partial p_s}\frac{\partial Q_i}{\partial q_r} - \frac{\partial P_i}{\partial q_r}\frac{\partial Q_i}{\partial p_s} =$$

$$\overset{\text{(from def of } X)}{=} \omega_{\alpha\beta} \frac{\partial X^\alpha}{\partial p_s}\frac{\partial X^\beta}{\partial q_r} \overset{\text{(from ***)}}{=} \delta_{sr} \quad \checkmark$$

8&9 - continued

We showed that $\alpha = \left(p_r - P_i \frac{\partial Q_i}{\partial q_r}\right) dq_r - P_i \frac{\partial Q_i}{\partial p_r} dp_r$

is exact. For this we used dq, dp basis.

But property of being exact does not depend on the basis.

So we know that

$$\underline{p\,dq - P\,dQ \quad \text{is exact}}$$

Conditions of exactness of the latter:

$$\left.\begin{aligned} \frac{\partial p_i[\,]}{\partial q_j} &= \frac{\partial p_j[\,]}{\partial q_i} \\ \frac{\partial P_i[\,]}{\partial Q_j} &= \frac{\partial P_j[\,]}{\partial Q_i} \end{aligned}\right\} \leftarrow \text{this the answer for 9}$$

$$\frac{\partial p_i[\,]}{\partial Q_j} = -\frac{\partial P_j[\,]}{\partial q_i} \quad \leftarrow \text{this was asked to prove in 8.}$$

Here $f[\,]$ denotes ~~fun~~ $f[\vec{q}, \vec{Q}]$

Note $\underset{\text{at const } \vec{Q}}{\frac{\partial (f[\vec{q}, \vec{Q}])}{\partial q_i}} \neq \underset{\text{at const } \vec{p}}{\frac{\partial f(\vec{q}, \vec{p})}{\partial q_i}}$!