④

General notes:

These are synonims: "symplectic flow" and "Hamiltonian flow"
These are synonims: "gradient flow" and "potential flow".

① condition for the vector field to have a potential is

Finding potentials

$$\partial_i (g_{jk} v^k) = \partial_j (g_{ik} v^k)$$

Because $g_{jk} = \delta_{jk}$, it simplifies to: $\underline{\underline{\partial_i v_j = \partial_j v_i}}$
we can identify $v_i = v^i$

So let us check: (a) $v = \begin{pmatrix} x \\ 1 \end{pmatrix}$; $\partial_x v_y \doteq \partial_y v_x$?

$$0 = 0 \quad \checkmark$$

(b) $v = \begin{pmatrix} -y \\ x \end{pmatrix}$; $\partial_x v_y \neq \partial_y v_x$, $1 \neq -1$ ✗

(c) All v_i are constants, so $\partial_i v_j = 0 = \partial_j v_i$, $\forall\, i,j$ ✓
[answer does not depend on a year]

(d) ~~Here we can check that it has~~
Because $v_i = x_i$, $\partial_i v_j = \partial_i x_j = 0$
(so it has potential) ✓

Now we will look for potential:

(a) $v_i = \partial_i V$; $\begin{cases} \partial_x V = x \\ \partial_y V = 1 \end{cases}$ → you can guess the answer

$$V = \frac{1}{2}x^2 + y$$

Systematic derivation is:

$$\left.\frac{\partial V}{\partial x}\right|_y = x \Rightarrow V = \frac{1}{2}x^2 + h(y) \; ; \left.\frac{\partial V}{\partial y}\right|_x = 1$$

On the other hand, $\left.\frac{\partial \bar{V}}{\partial y}\right|_x = \left.\frac{\partial(\frac{1}{2}x^2 + h(y))}{\partial y}\right|_x = \frac{dh}{dy}$

Hence $\frac{dh}{dy} = 1 \Rightarrow h = y \; (+\text{const}) \Rightarrow \bar{V} = \frac{1}{2}x^2 + y$

(c)

$$\begin{aligned} \partial_x \bar{V} &= 2 \\ \partial_y \bar{V} &= 0 \\ \partial_z \bar{V} &= 1 \\ \partial_t \bar{V} &= 4 \end{aligned} \Rightarrow \bar{V} = 2x + 0y + 1\cdot z + 4t$$

(this part of the answer depends on a year)

(d) $\cancel{\partial_i \bar{V} = x}$ $\frac{\partial \bar{V}}{\partial x^i} = x_i$ for $i = \overline{1,4}$ $\Rightarrow$ $\boxed{\bar{V} = \sum_{i=1}^{4} \frac{1}{2} x_i^2}$

Note: only from $\frac{\partial \bar{V}}{\partial x_1} = x_1$ we do not conclude that $\bar{V} = \frac{1}{2}x_1^2$. We can conclud that

$$\bar{V} = \frac{1}{2}x_1^2 + h(x_2, x_3, x_4)$$ and then constrain

h from $\frac{\partial \bar{V}}{\partial x_2} = x_2, \ldots$

① Finding Hamiltonians

Condition for the vector field to be Hamiltonian is:

$$\partial_i (\omega_{jk} V^k) = \partial_j (\omega_{ik} V^k)$$

in 2d case it simplifies:

$$\partial_1 V^1 = -\partial_2 V^2 \Rightarrow \operatorname{div} \vec{V} = 0 \quad \text{(Liouville theorem!)}$$

~~in 4d case it would be~~

Let me note that $\operatorname{div}\vec{V} = 0$ should be true in any even dimension, but it is not enough to conclude that $V^i = \omega^{ij}\partial_j H$ (only in 2d!)

① Finding Hamiltonians, continued

(a) $\operatorname{div}\vec{v} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(1) = 1$ ✗

(b) $\operatorname{div}\vec{v} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$ ✓

(c) Because all entries are constants,

$$\partial_i(\omega_{jk}v^k) = 0 = \partial_j(\omega_{ik}v^k) \quad ✓$$

(d) $\operatorname{div}\vec{v} = \sum_{i=1}^{4} \frac{\partial}{\partial x_i} x_i = 4$ ✗

(it was enough to check that div $\neq 0$).

Finding Hamiltonian:

(b): $v^1 = (\omega^{-1})^{12} \frac{\partial H}{\partial x^2}\Big|_{x^1} \Rightarrow v_x = \frac{\partial H}{\partial y}\Big|_x = -y$ from $v = \{-y, x\}$

$$v_y = -\frac{\partial H}{\partial x}\Big|_y = x$$

$$\frac{\partial H}{\partial y}\Big|_x = -y \;;\; -\frac{\partial H}{\partial x}\Big|_y = x \Rightarrow H = -\frac{1}{2}(x^2+y^2)$$

You could also simply recall that the vector flow $\dot{x} = y$, $\dot{y} = -x$ is the one of harmonic oscillator.

(c)
$$\left.\begin{aligned} 2 = v_x &= \frac{\partial H}{\partial z} \\ 0 = v_y &= \frac{\partial H}{\partial t} \\ 1 = v_z &= -\frac{\partial H}{\partial x} \\ 4 = v_t &= -\frac{\partial H}{\partial y} \end{aligned}\right\} \Rightarrow \underline{H = 2z + 0 \cdot t - 1 \cdot x - 4 \cdot y}$$

(2) We recall that $\vec{M} = \vec{x} \times \vec{p}$, or

in components $M_i = \epsilon_{ijk}\, x_j\, p_k$

Our phase space is 6-dimensional: $\{x, y, z, p_x, p_y, p_z\}$

The Hamiltonian vector ~~flow~~ field is given by $v_H = \left\{ \frac{\partial H}{\partial p_x}, \frac{\partial H}{\partial p_y}, \frac{\partial H}{\partial p_z}, -\frac{\partial H}{\partial x}, -\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial z} \right\}$

It generates vector flow according to equations:

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{p}_x = -\frac{\partial H}{\partial x}$$
$$\dot{y} = \frac{\partial H}{\partial p_y} \qquad \dot{p}_y = -\frac{\partial H}{\partial y}$$
$$\dot{z} = \frac{\partial H}{\partial p_z} \qquad \dot{p}_z = -\frac{\partial H}{\partial z}$$

, where $H = M_i$. Here e.g. $\dot{x} = \frac{dx}{d\tau}$, and τ is a "time" associated with the flow. It is not the physical time ticking in our clocks.

$$-\frac{\partial M_i}{\partial x_j} = -\epsilon_{ijk}\, p_k \ ; \ +\frac{\partial M_i}{\partial p_j} = +\epsilon_{ikj}\, x_k = -\epsilon_{ijk}\, x_k$$

$$v_{M_x} = \{0, -z, y, 0, -p_z, p_y\}$$

$$v_{M_y} = \{z, 0, -x, p_z, 0, -p_x\}$$

$$v_{M_z} = \{-y, x, 0, -p_y, p_x, 0\}$$

Consider for instance v_{M_x}. "Equations of motion" are

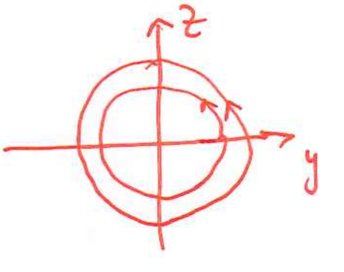
$$\dot{x} = 0 \qquad \dot{p}_x = 0$$
$$\dot{y} = -z \qquad \dot{p}_y = -p_z$$
$$\dot{z} = y \qquad \dot{p}_z = p_y$$

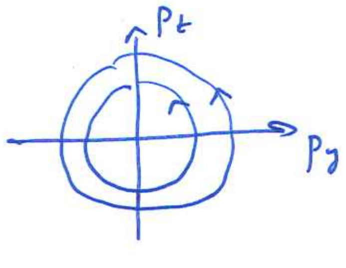
$\therefore$ $\left.\begin{array}{l} \dot{y} = -z \\ \dot{z} = y \end{array}\right\}$ → decouples from the rest

$\therefore$ $\left.\begin{array}{l} \dot{p}_y = -p_z \\ \dot{p}_z = p_y \end{array}\right\}$ → decouples from the rest

$\dot{x} = 0 \Rightarrow x = \text{const}$

$\dot{p}_x = 0 \Rightarrow p_x = \text{const}$





<u>Note:</u> M_x is a conserved quantity in systems which are invariant under rotations around x-axis. Hamiltonian flow generated by M_x, as we saw, precisely generates rotations around x-axis.

This is not a coincidense. ~~If~~ Hamiltonian flow of any function F describes precisely the transformations under which F is conserved

M_y, M_z are treated in the same way.

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(3) Meaning of the question. Consider a 2d vector flow equations of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

, with A diagonalizable and $\underline{\det A \neq 0.}$

For which A this flow is Hamiltonian? Then describe qualitatively different situations.

In 2d it is enough to check that $\operatorname{div} \vec{v}_H = 0$ (Liouville theorem)

$$V_x = A_{11} x + A_{12} y \; ; \; V_y = A_{21} x + A_{22} y$$

$$\operatorname{div} \vec{v}_H = \frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y = A_{11} + A_{22} = \operatorname{tr} A$$

So demand is $\boxed{\operatorname{tr} A = 0}$ $\Rightarrow$ $\lambda_1 + \lambda_2 = 0$ eigenvalues.

There are only qualitatively 2 different cases

$$\lambda_1 \cdot \lambda_2 = \det A > 0$$

$$\lambda_1 \cdot \lambda_2 = \det A < 0$$

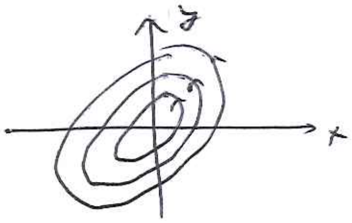
For $\lambda_1 \cdot \lambda_2 > 0 \Rightarrow \lambda_1 = i \sqrt{|\det A|}$, $\lambda_2 = -i \sqrt{|\det A|}$ $\to$ critical point of type center (like harmonic oscillator)

~~in gen.~~

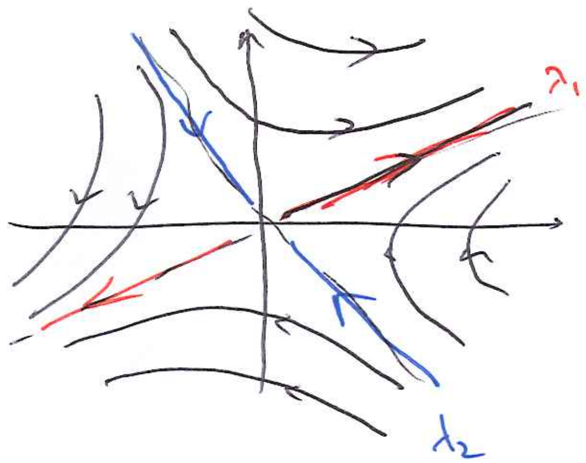
(direction of rotation can be any)

Generically type center is a collection of ellipses

Tutorial 3 (6)
solutions
part II



For $\lambda_1\lambda_2 < 0 \Rightarrow \lambda_1 = \sqrt{|\det A|}$, $\lambda_2 = -\sqrt{|\det A|}$ - this is saddle point



(4) As we learned from problem (1), ~~vector~~ constant vector field is of gradient and symplectic type.

Now generically, we have to satisfy:

$$v_x = \frac{\partial V}{\partial x} = \frac{\partial \mathcal{H}}{\partial y}$$

$$v_y = \frac{\partial V}{\partial y} = -\frac{\partial \mathcal{H}}{\partial x}$$

If you recall complex analysis, these are conditions for function $\omega = V + i\mathcal{H}$ to be an analytic function of $z = x + iy$:

$$\frac{\partial V(x+iy) + i\mathcal{H}(x+iy)}{\partial x} = -i\frac{\partial V(x+iy) + i\mathcal{H}(x+iy)}{\partial y} \Rightarrow \begin{array}{l} \frac{\partial V}{\partial x} = \frac{\partial \mathcal{H}}{\partial y} \\ \frac{\partial V}{\partial y} = -\frac{\partial \mathcal{H}}{\partial x} \end{array}$$

4 - continued

The same is true for $\{\vartheta_x, \vartheta_y\}$

Tutorial 3 (7)

solutions part II

From ϑ being gradient: $\partial_y \vartheta_x = \partial_x \vartheta_y$

ϑ being symplectic: $\partial_x \vartheta_x = -\partial_y \vartheta_y$

So condition: combination $\vartheta_x - i\vartheta_y$ should be an analytic function of $x+iy$.

(5). The way to do this problem is to write a covariant form for gradient form:

$$\dot{x}^i = g^{ij} \frac{\partial V}{\partial x^j}$$

~~g^{ij} is~~ $g^{ij} \equiv (g^{-1})^{ij}$, and g_{ij} is a metric of the space.

You should know by heart metric {when derivin Lagrangian, you did this} in polar coordinates $\{r, \varphi\}$: $g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

in spherical coordinates $\{r, \theta, \varphi\}$: $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

Or derive from

(*) $$\underset{\substack{\uparrow \\ \text{metric in the frame} \\ \text{defined by } y=y(x)}}{g^{ij}_{\text{"}y\text{"}}} = \underset{\substack{\uparrow \quad \uparrow \\ \text{Descartian coordinates}}}{\frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta}} \underset{\substack{\uparrow \\ \text{(inverse) metric in} \\ \text{Desc. coordinates}}}{\delta^{\alpha\beta}} = \sum_{k=1}^{\overset{\text{dimension}}{n}} \frac{\partial y^i}{\partial x^k} \frac{\partial y^j}{\partial x^k} = \vec{\nabla} y^i \cdot \vec{\nabla} y^j$$

~~Polar coordinates: $\frac{\partial r}{\partial x^i} = \frac{x^i}{r}$,~~

Or: (**) $$g^{\text{"}y\text{"}}_{ij} = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \delta_{\alpha\beta} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j}$$

~~g_{12} For complex coordinates $g^{11} = \partial(x+iy$~~

For complex coordinates:

$$z = x + iy$$
$$\bar{z} = x - iy$$

From *:

$$g^{zz} = \frac{\partial z}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial z}{\partial y} = 0$$

$$g^{\bar{z}\bar{z}} = 0$$

$$g^{z\bar{z}} = \frac{\partial z}{\partial x}\frac{\partial \bar{z}}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial \bar{z}}{\partial y} = 1 + 1 = 2$$

$$g^{\bar{z}z} = 2$$

$$g^{ij} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Hence gradient flow equations are written as:

$$\dot{r} = \frac{\partial V}{\partial r}$$
$$\dot{\varphi} = \frac{1}{r^2}\frac{\partial V}{\partial \varphi}$$

↑ from $g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$

$$\dot{r} = \frac{\partial V}{\partial r}$$
$$\dot{\theta} = \frac{1}{r^2}\frac{\partial V}{\partial \theta}$$
$$\dot{\varphi} = \frac{1}{r^2 \sin^2\theta}\frac{\partial V}{\partial \varphi}$$

from $g^{ij} = \begin{pmatrix} 1 & & \\ & r^{-2} & \\ & & r^{-2}\sin^{-2}\theta \end{pmatrix}$

$$\dot{z} = 2\frac{\partial V}{\partial \bar{z}}$$
$$\dot{\bar{z}} = 2\frac{\partial V}{\partial z}$$

from $g^{ij} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$