Solutions

part I

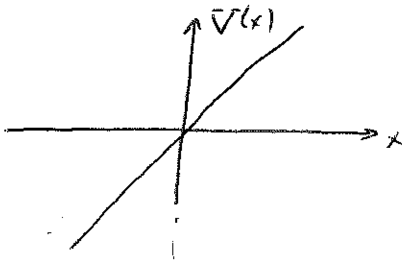
$n=1$ (one-dimensional case)

In the equation $\dot{x} = V(x)$, if $V > 0$, then $\dot{x} > 0$ => particle moves to the right

$V < 0$, then $\dot{x} < 0$ => particle moves to the left

$V = 0$ => particle does not move. it is ~~a~~ critical point.

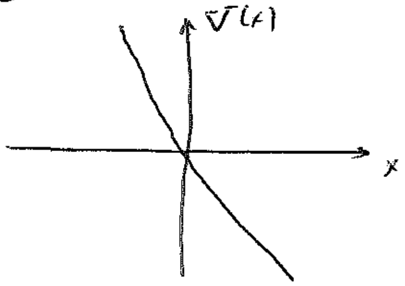
(a) $\dot{x} = 2x$



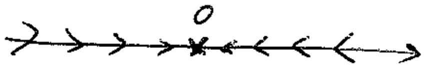
phase portrait:

(arrows increase to emphasize that particle moves faster far from origin then near the origin)

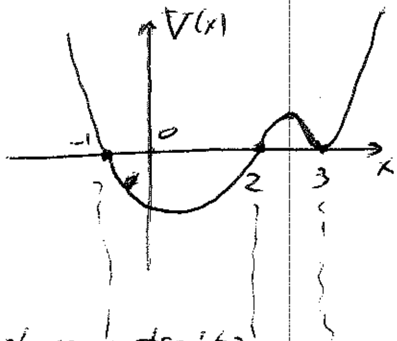
(b) $\dot{x} = -2x$



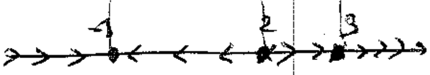
phase portrait:



(c) $\dot{x} = (x+1)(x-2)(x-3)^2$



phase portrait:



Note the structure of the vector flow near $x=3$. It is different because $x=3$ is double zero.

$n=2$ (Two-dimensional case), linear systems.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}A_{11}x + A_{12}y\\ A_{21}x + A_{22}y\end{pmatrix}$$

(a) $A = \begin{pmatrix}1&0\\0&2\end{pmatrix}$; ~~we just solve~~ Fast way: $\dot{x} = x$, $\dot{y} = 2y$ => $x = x_0 e^{t}$, $y = y_0 e^{2t}$ (~~we could also make a canonical approach: - find eigenvalue~~)

Canonical way (in this case slow): consider an ansatz $\vec{X}(t) = \vec{c}_i e^{\lambda_i t}$; $\vec{x} = \begin{pmatrix}x\\y\end{pmatrix}$

$\dot{\vec{x}} = A\vec{x}$ => $\vec{c}_i \lambda_i = A\vec{c}$ → eigenvalue problem

$\vec{c}_i = \begin{pmatrix}c_{1i}\\c_{2i}\end{pmatrix}$ constant vector

solution I: $\vec{c} = \begin{pmatrix}1\\0\end{pmatrix}$; $\lambda = 1$; II: $\vec{c} = \begin{pmatrix}0\\1\end{pmatrix}$; $\lambda = 2$

n=2, (a) - continued

~~Here~~. A:

Tutorial 3 (2)
solutions
part II

A generic solution is a linear combination of ~~the original~~ two solutions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{x}(t) = \underset{\text{constant}}{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + \underset{\text{constant}}{B} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} Ae^{t} \\ Be^{2t} \end{pmatrix}.$$

It is suggestive to denote

$A = x_0$ (x at $t=0$)

$B = y_0$ (y at $t=0$)

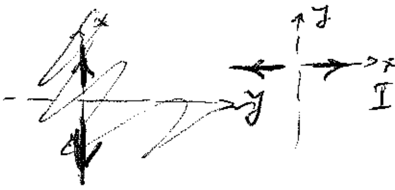
Conclusion: $x = x_0 e^{t}$
$y = y_0 e^{2t}$

A set $\{x_0, y_0\}$ parameterises different curves. However, because we cosider $t \in (-\infty, +\infty)$, ~~we can~~ and want to plot curves for all values of t's, it is harmless to make a shift: $t = \tau - \log x_0$ an ~~to~~ plot curves for all values of τ:

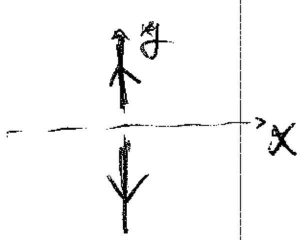
$$x = e^{\tau}$$
$$y = \left(y_0/x_0^2\right) \cdot e^{2\tau}$$

So, in fact, there is only one parameter: y_0/x_0^2

If $y_0 = 0$, $\Rightarrow$
$\Rightarrow y = 0 \Rightarrow$

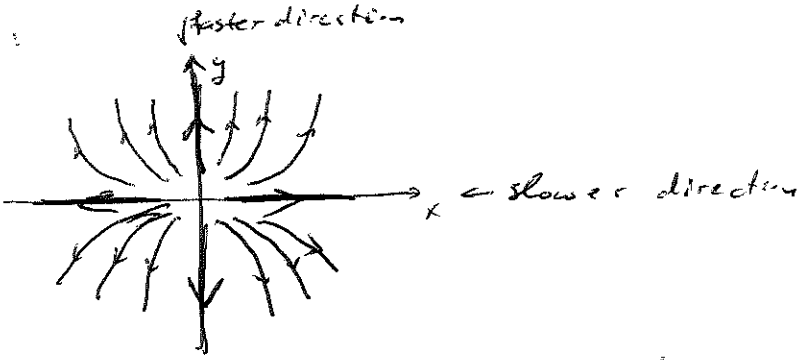


If $x_0 = 0$, it is the best to come back to $x = x_0 e^{t} = 0$
$y = y_0 e^{2t} \Rightarrow$



From this asymptotes we complete phase portrait using common sense:

[This critical point is called unstable <u>node</u>]

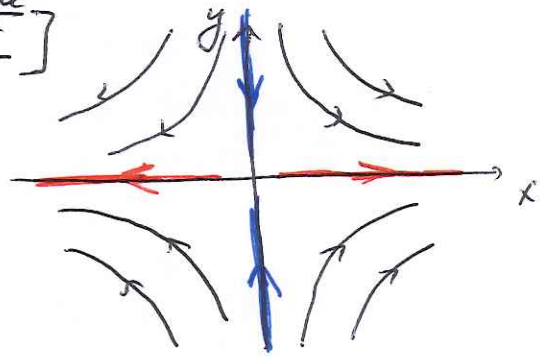


You can also ~~solv~~ exclude t from: $\begin{cases} x = x_0 e^{t} \\ y = y_0 e^{2t} \end{cases} \Rightarrow \boxed{y = \frac{y_0}{x_0^2} x^2}$

~~So the~~ So each curve  is a plot of parabola $y = \alpha x^2$

$\underline{n=2}$ (b) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{array}{l} x = x_0 e^{t} \\ y = y_0 e^{-t} \end{array}$

[saddle point]



red — : solution $x = x_0 e^{t}$, $y = 0$

blue — : solution $x = 0$, $y = y_0 e^{-t}$

— : any other solution

In this case you can also solve explicitly $y = y(x)$:

$$y = \frac{y_0 x_0}{x}$$

Each line  is a plot of $y = \frac{\alpha}{x}$.

(c) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$; $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$

Use canonical approach: $\vec{x} = \vec{c}\, e^{\lambda t}$, $\vec{c}$ constant (does not depend on time)

$$\dot{\vec{x}} = A\vec{x} \Rightarrow \lambda \vec{c} = A\vec{c}$$

Solution of the eigenvalue proplem: $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$; $\vec{c}_1 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$, $\vec{c}_2 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$

[If you do some algorithm to find solution, always check the result!!! (by substituting into original equation)

Here $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \pm 1 \end{pmatrix} = \begin{pmatrix} \pm 2 \\ \sqrt{2} \end{pmatrix} = \pm\sqrt{2} \begin{pmatrix} \sqrt{2} \\ \pm 1 \end{pmatrix}$ ✓]

short cut: $\left.\begin{array}{l} \operatorname{tr} A = \lambda_1 + \lambda_2 = 0 \\ \det A = \lambda_1 \lambda_2 = -2 \end{array}\right\} \Rightarrow \lambda = \pm\sqrt{2}$

eigenvectors can be guessed and then checked

Generic solution: $\vec{x} = A \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{\sqrt{2}t} + B \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-\sqrt{2}t}$

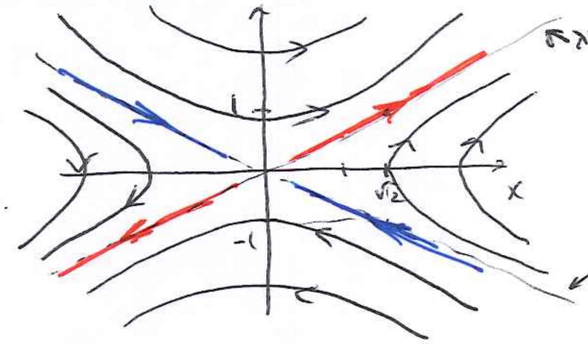
Asymptotes: $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{\sqrt{2}t}$; $\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-\sqrt{2}t}$

n=2, (c) - cont-ed

Tutorial 3 (4)

solution

part I



[saddle point]

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n=2, (d)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 2\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \quad ; \text{ here } \quad \left.\begin{array}{l} tr\,A = 0 \\ det\,A = 2\end{array}\right\} \Rightarrow \begin{array}{l}\lambda_1 = i\sqrt{2}\\ \lambda_2 = -i\sqrt{2} = \overline{\lambda_1}\end{array}$$

Complex eigenvalues always means rotations in the phase portrait. If matrix A has only real coefficients, then complex eigenvalues always come in the conjugated pairs

$$\vec{x} = A\begin{pmatrix}-i\sqrt{2}\\ 1\end{pmatrix} e^{i\sqrt{2}t} + \bar{A}\begin{pmatrix}+i\sqrt{2}\\ 1\end{pmatrix} e^{-i\sqrt{2}t} = Re\left[A\begin{pmatrix}-i\sqrt{2}\\ 1\end{pmatrix} e^{i\sqrt{2}t}\right]$$

complex conjugated expression

$A = a + ib$

real real

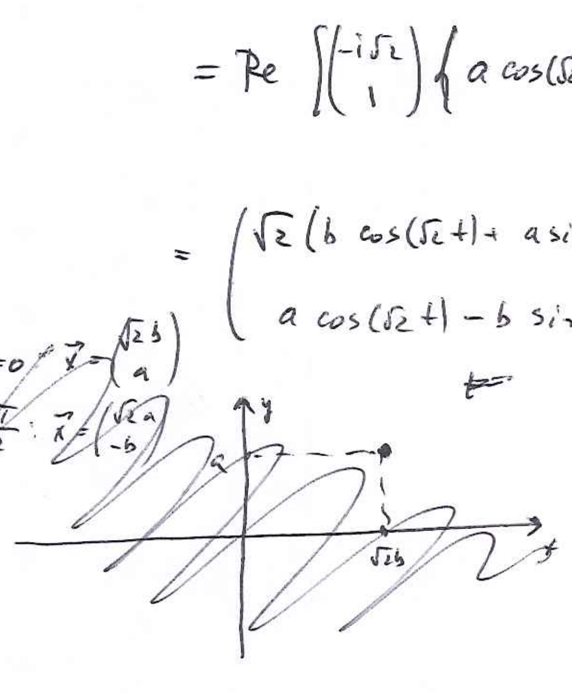
$$\vec{x} = Re\left[(a+ib)\begin{pmatrix}-i\sqrt{2}\\ 1\end{pmatrix}\cdot\left(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)\right)\right] =$$

$$= Re\left[\begin{pmatrix}-i\sqrt{2}\\ 1\end{pmatrix}\left\{a\cos(\sqrt{2}t) - b\sin(\sqrt{2}t) + i\left(b\cos(\sqrt{2}t) + a\sin(\sqrt{2}t)\right)\right\}\right] =$$

$$= \begin{pmatrix}\sqrt{2}\left(b\cos(\sqrt{2}t) + a\sin(\sqrt{2}t)\right)\\ a\cos(\sqrt{2}t) - b\sin(\sqrt{2}t)\end{pmatrix} \leftarrow \text{parameterisation of an ellips}$$

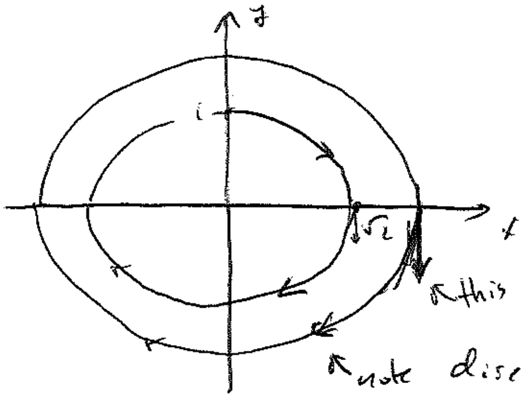
$t=0$: $\vec{x} = \begin{pmatrix}\sqrt{2}b\\ a\end{pmatrix}$

$\sqrt{2}t = \frac{\pi}{2}$: $\vec{x} = \begin{pmatrix}\sqrt{2}a\\ -b\end{pmatrix}$



let $b = R\cos\varphi$ (polar parameterisation)
$a = \pm R\sin\varphi$

$$\vec{x} = \begin{pmatrix}\sqrt{2}R\cdot\cos(\sqrt{2}t \mp \varphi)\\ -R\sin(\sqrt{2}t \mp \varphi)\end{pmatrix}$$



↖ this tangent vector is given by $A\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -x \end{pmatrix}$

↖ note direction of rotation. it follows from sign in $\begin{pmatrix} \sqrt{2}\cos \\ -\sin \end{pmatrix}$,
but also you can check: $\boxed{\begin{matrix} \dot{x}\cancel{6x} = 0 \\ \dot{y} = -x \end{matrix} \text{ for } y=0 \text{ points}}$

Note: equations $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ are scale invariant.

This means that variables $X = \lambda x$, $Y = \lambda y$ for any $\lambda \in \mathbb{R}$ obey exactly the same equations as x, y. $\Rightarrow$

$\Rightarrow$ All ellipses are of the same shape and orientation.

$n=2$, (e)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

~~Ed~~ Associated eigenproblem:
$$\begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix} \vec{c} = \lambda \vec{c}$$

$$\mathrm{tr}\, A = 0 \rightarrow \lambda_1 + \lambda_2 = 0$$
$$\det A = -\cos^2 2\phi - \sin^2 2\phi = -1 \Rightarrow \lambda_1 \lambda_2 = -1 \quad \Rightarrow \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -1 \end{matrix} \quad \left[\text{should be a saddle point} \right]$$

$$\vec{c} = \begin{pmatrix} a \\ b \end{pmatrix}; \; \lambda = \pm 1: \quad \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ -\sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix} \quad (*)$$

$$a = \frac{(\sin 2\phi) \cdot b}{(\cos 2\phi) \mp 1}; \quad \text{check it:} \; -\sin 2\phi \cdot \frac{\sin 2\phi \cdot b}{\cos 2\phi \mp 1} - \cos 2\phi\, b = \pm b$$

$$-\sin^2 2\phi = (\cos 2\phi \pm 1)(\cos 2\phi \mp 1) =$$
$$= \cos^2 2\phi - 1 = -\sin^2 2\phi \; \checkmark$$

$$\frac{\sin(2\phi)}{\cos(2\phi) \mp 1} = \frac{2 \sin\phi \cos\phi}{\cos^2\phi - \sin^2\phi \mp 1} = \begin{cases} -\cot\phi, & \text{"}-\text{" case} \\ +\tan\phi, & \text{"}+\text{" case} \end{cases}$$

So: $\lambda = 1$: $\vec{e}_1 = \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix}$: check: $\begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ -\sin 2\varphi & -\cos 2\varphi \end{pmatrix} \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} =$

$= \begin{pmatrix} -\sin\varphi \\ -\cos\varphi \end{pmatrix}$ ← was mistake !

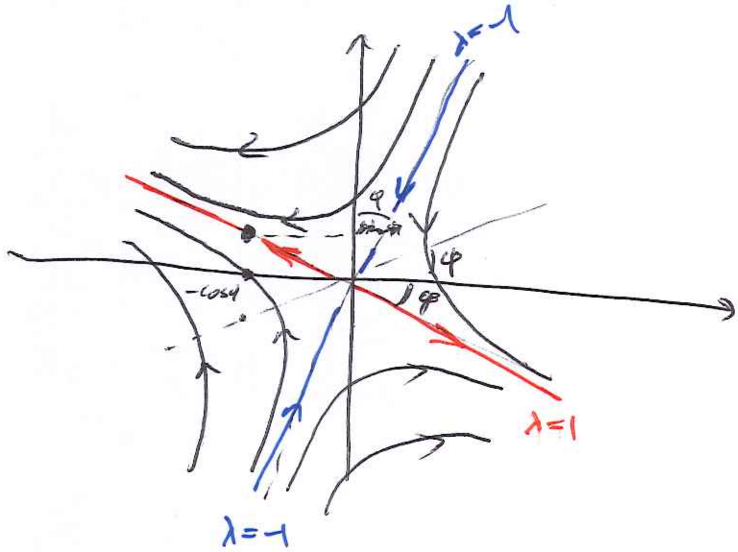
should be $\boxed{\lambda = -1 \,;\ \vec{e}_{-1} = \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix}}$

Then $\boxed{\lambda = 1 \,;\ \vec{e}_1 = \begin{pmatrix} -\cos\varphi \\ \sin\varphi \end{pmatrix}}$

check: $\begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ -\sin 2\varphi & -\cos 2\varphi \end{pmatrix} \begin{pmatrix} -\cos\varphi \\ \sin\varphi \end{pmatrix} =$

$= \begin{pmatrix} -\cos\varphi \\ \sin\varphi \end{pmatrix}$ ✓

the mistake is: "−" case is misinterpreted it is "upper sign case" in equation (*)
The upper sign ~~should~~ is ~~be~~ "+" in (*)



Interpretation:
saddle point from $n=2$, (b) case rotated clockwise by an angle φ.

n=2, generic

(a) $\mathcal{H} = \frac{p^2}{2m}$ => Equations of motion:

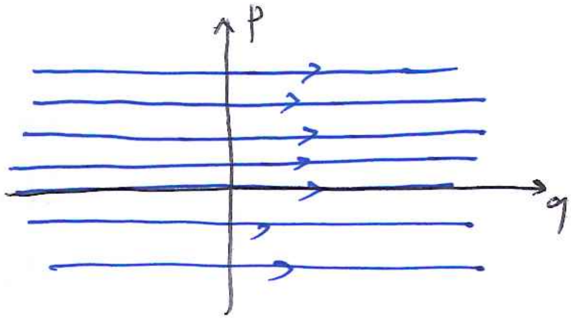
$$\dot{q} = p/m$$

$$\dot{p} = 0$$

solution $p = \text{const}$

$$q = \frac{pt}{m} + q_0$$

The plot of phase portrait is:



(b) $\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2$ =>

=> Equations of motion: $\dot{q} = p/m$

$$\dot{p} = -m\omega^2 q$$

solution follows the same steps as in n=2, d.

In fact, we can rewrite:
$$\frac{dq}{d(\omega t)} = \left(\frac{p}{\omega m}\right); \quad \frac{d\left(\frac{p}{\omega m}\right)}{d(\omega t)} = -q$$

$\cancel{\frac{dp/\omega m}{d(\omega t)} =}$

In the last normalisation this should be just normal circles:

$$q = A\cos(\omega t + \varphi) \;; \; \left(\frac{p}{\omega m}\right) = A \sin(\omega t + \varphi)$$

$\cancel{\dot{p} =}$

$$\Downarrow$$

$$p = A \cdot m\omega \sin(\omega t + \varphi)$$

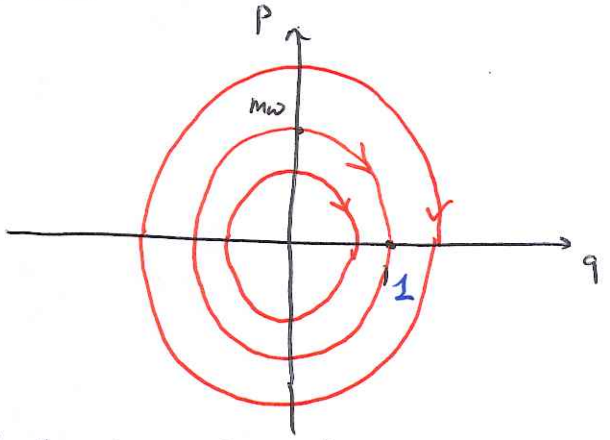
[Here you should spend 30 sec and check that it is a solution] n=2, generic, cont-ed

If you spent 30 sec as asked, you will find a sign mistake!

Correct solution is:

$$q = A \sin(\omega t + \varphi)$$
$$p = A \cdot m\omega \cdot \cos(\omega t + \varphi)$$

Phase portrait:



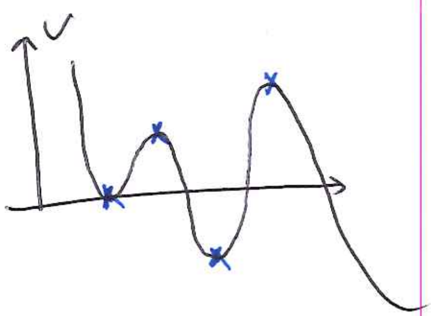
✓ Check that direction of movement is ok

✓ Since equations of motion are scale invariant, all ellipses are of the same shape.

n=2, generic

(c) $\mathcal{H} = \frac{p^2}{2m} + V$; Equations of motion:

$$\dot{q} = p/m$$
$$\dot{p} = -V'(x)$$



This is a nonlinear system.

Approach is: • find critical points
• then linearise near each critical point.

Here critical points are: $p/m = 0 \Rightarrow p = 0$
$+V' = 0 \rightarrow$ extrema of potential (X on the picture)

Tutorial 3 solutions (9) part I

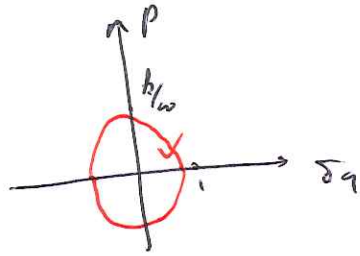
Near each critical point: $-V'(q) = \underbrace{-V'(q_0)}_{= 0} + \underbrace{\delta q}_{= x - x_0} \cdot (-V''(q))$

If $V'' > 0$ (minimum), equations are: $\dot{\delta q} = p/m$

$\dot{p} = -\underbrace{k}_{= V''} \delta q \ , \ k > 0$

By parametrising: $k = m\omega^2$, we

get precisely harmonic oscillator, discussed in case (b).



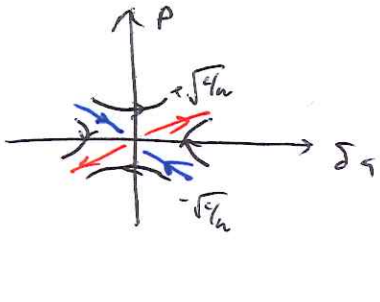
If $V'' < 0$ (maximum), equations are: $\dot{\delta q} = p/m$

$\dot{p} = c \cdot \delta q \ , \ c > 0$

Hence $A = \begin{pmatrix} 0 & 1/m \\ c & 0 \end{pmatrix}$; $\operatorname{tr} A = 0 \Rightarrow \lambda_1 + \lambda_2 = 0$

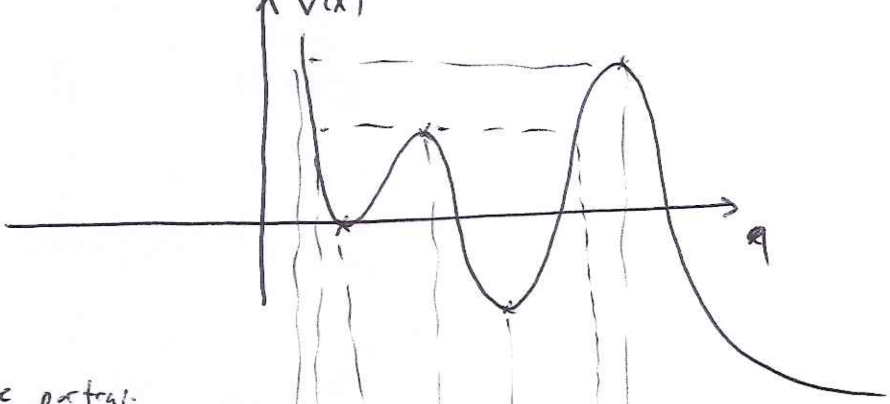
$\det A = -c/m \Rightarrow \lambda_1 = \sqrt{c/m}$, $\lambda_2 = -\sqrt{c/m}$ (saddle point)

eigen vectors: $\cancel{\begin{pmatrix} m \\ \pm c \end{pmatrix}}$ $\begin{pmatrix} 1 \\ \pm\sqrt{mc} \end{pmatrix}$

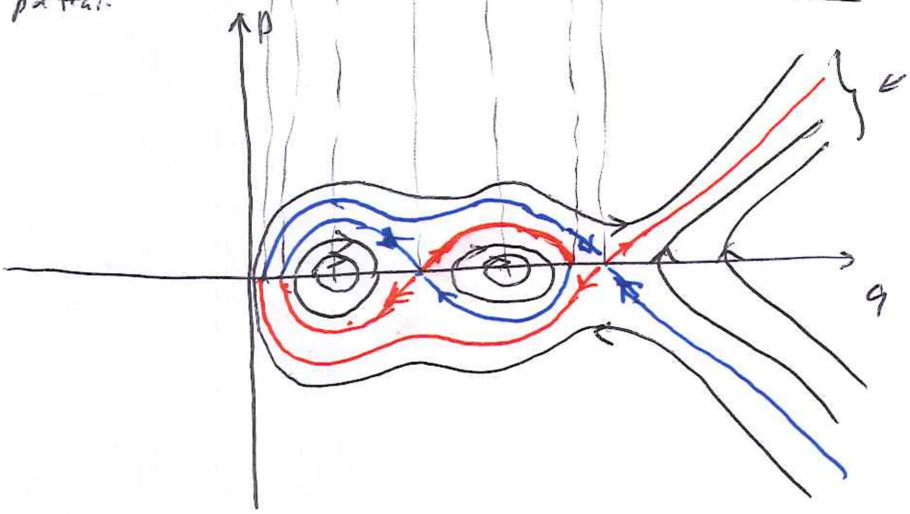


we now only need to combine together:

- All trajectories should be closed because of the energy conservation.
- Phase portrait should be symmetric with respect to $p \leftrightarrow -p$ (because Hamiltonian and equations of motion are)



Phase portrait



← this behavior is nonlinear, it depends on how $V(x) \to \infty$, when $x \to \infty$.