Levi-Civita symbol in 3D

ε_{ijk} is fully antisymmetric rank 3 tensor

- rank 3 means – 3 indices
- fully antisymmetric means:

$$\varepsilon_{ijk} \overset{(1)}{=} -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$$

exchange of 1st and 2nd; exchange of 2nd & 3rd; exchange of 1st & 3rd

- Show that $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$ (it does not change sign under cyclic permutations)

- i, j, k are indices. They stand for some explicit numbers from set $\{1,2,3\}$ (because we live in 3 dimensions)

So there are $3 \times 3 \times 3 = 27$ components in tensor ε

(similarly as there are $3 \times 3 = 9$ components M_{ij} in any 3×3 matrix M, which can be thought as rank 2 tensor)

- But most of the components are <u>zero</u>!

E.g: $\varepsilon_{112} \underset{\text{from (1)}}{=} -\varepsilon_{112} \Rightarrow \varepsilon_{112} = 0$

Only if $\begin{cases} i \neq j \\ i \neq k \\ j \neq k \end{cases}$ (all are distinct), ε_{ijk} may be non-zero.

But there are only 6 such possibilities:

$$\varepsilon_{123} ; \ \varepsilon_{213} ; \ \varepsilon_{321} ; \ \varepsilon_{231} ; \ \varepsilon_{312} ; \ \varepsilon_{132}$$

- We fully determine ε_{ijk} by setting $\varepsilon_{123} = 1$
- <u>Show that</u>

$\varepsilon_{213} = -1 \quad \varepsilon_{231} = +1 \quad \varepsilon_{132} = -1$

$\varepsilon_{321} = -1 \quad \varepsilon_{312} = +1$

(all the rest are zero due to antisymmetry)

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- Notation like $\varepsilon_{ijk} V^k$ means $\sum_{k=1}^{3} \varepsilon_{ijk} V^k$ [in this text I won't distinguish V^k and V_k]
- Note that "k" is a mute index of summation (like an internal variable in the definition of function/procedure in programming language).

So $\frac{\partial}{\partial V^\alpha} (\varepsilon_{ijk} V^k) \overset{(2)}{=} \varepsilon_{ij\alpha}$, although there is no α explicitly in the expession $\varepsilon_{ijk} V^k$

Explanation: $\varepsilon_{ijk} V^k = \varepsilon_{ij1} V^1 + \varepsilon_{ij2} V^2 + \varepsilon_{ij3} V^3$, α can ~~mak~~ be either 1 or 2 or 3, and nothing else

If $\alpha = 1$: $\frac{\partial}{\partial V^1} (\varepsilon_{ij1} V^1 + \varepsilon_{ij2} V^2 + \varepsilon_{ij3} V^3) = \varepsilon_{ij1}$

$\alpha = 2$: $\frac{\partial}{\partial V^2} (\varepsilon_{ij1} V^1 + \ldots \ldots) = \varepsilon_{ij2}$

$\alpha = 3$: $\frac{\partial}{\partial V^3} (\ldots \ldots) = \varepsilon_{ij3}$

And in short, these 3 equations are denoted by (2).

- In general, $\frac{\partial}{\partial V^i} V^k = \left\{ \begin{matrix} 1 & \text{if } i=k \\ 0 & \text{if } i \neq k \end{matrix} \right\} = \delta_i^{\ k}$ ← Kronecker delta.

- There is nice interplay between ε_{ijk} and exterior product of vectors

let $\vec{v}, \vec{w}$ be two vectors, and $\boxed{\vec{u} \equiv \vec{v} \times \vec{w}}$

~~The~~ Check that $\boxed{u_i = \epsilon_{ijk} V_j W_k}$ (write down in components e.g

$$u_1 = \epsilon_{1jk} V_j W_k =$$

$$= \overset{=0}{\epsilon_{111}} V_1 W_1 + \overset{=0}{\epsilon_{112}} V_1 W_2 +$$

$$+ \overset{=0}{\epsilon_{121}} V_2 W_1 + \overset{=0}{\epsilon_{122}} V_2 W_2$$

$$+ \overset{=1}{\epsilon_{123}} V_2 W_3 + \ldots =$$

$$= V_2 W_3 - V_3 W_2 \text{)}$$

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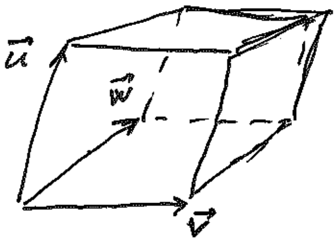
- Now it is clear: $\vec{u}\cdot(\vec{v}\times\vec{w}) = \epsilon_{ijk}\, u_i v_j w_k$

  and you can derive from $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$:

  $$\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v}) \quad \text{(\underline{do this!})}$$

  Btw $\vec{u}\cdot(\vec{v}\times\vec{w})$ is the volume of the following parallellogram:



- Other properties: $\vec{u}\times(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{u}\cdot\vec{w}) - \vec{w}\,(\vec{u}\cdot\vec{v})$

  [It is mnemonically known as "BAC − CAB" f-la due to $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$]

  We can prove it using Levi-Civita

  $$\left(\vec{u}\times(\vec{v}\times\vec{w})\right)_i = \epsilon_{ijk}\, u_j\, (\vec{v}\times\vec{w})_k = \epsilon_{ijk}\, u_j\, \epsilon_{krl}\, v_r w_l$$

  <u>prove it!</u>

  $$\epsilon_{ijk}\,\epsilon_{krl} = \epsilon_{ijk}\,\epsilon_{rlk} = \delta_{ir}\delta_{jl} - \delta_{il}\delta_{jr}$$

  summation over k;
  (actually, only at most one term is non-zero in this summation, when $k\neq i$, $k\neq j$, $k\neq r$, $k\neq l$)

  $$\left(\vec{u}\times(\vec{v}\times\vec{w})\right)_i = \epsilon_{ijk}\,\epsilon_{rlk}\, u_j v_r w_l =$$

  $$= \left(\delta_{ir}\delta_{jl} - \delta_{il}\delta_{jr}\right) u_j v_r w_l =$$

  $$= \delta_{ir} v_r\, \delta_{jl} u_j w_l - \left(\delta_{jr} v_r u_j\right)\delta_{il} w_l =$$

  $$= v_i\,(\vec{u}\cdot\vec{w}) - w_i\,(\vec{v}\cdot\vec{u})$$

- $$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\,\vec{C})(\vec{B}\,\vec{D}) - (\vec{A}\,\vec{B})(\vec{B}\,\vec{C})$$

  <u>Prove it!</u> (Using $\epsilon_{ijk}\,\epsilon_{rlk} = \delta_{ir}\delta_{jl} - \delta_{il}\delta_{jr}$)

- Levi-Civita establishes isomorphism between vectors and rank-2 antisymmetric tensors in 3D:

$$(*) \quad M_{ij} = \epsilon_{ijk} V_k \;;$$
$$(**) \quad V_i = \tfrac{1}{2}\epsilon_{ijk} M_{jk} \;;$$

(check that (*) and (**) ~~are~~ ~~equal~~ ~~equivalent~~ follow from one another)

Explicitly: $\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \longleftrightarrow M = \begin{pmatrix} 0 & V_1 & -V_2 \\ -V_1 & 0 & V_3 \\ +V_2 & -V_3 & 0 \end{pmatrix}$

Note that $\det M = 0$; ~~and $\frac{1}{2}\mathrm{tr}\, MM^T = \epsilon_{ij} M_{ii}$~~
$\mathrm{tr}\, M = 0$

$$\cancel{\mathrm{tr}\, M^2 = M_{ij}} \quad \mathrm{tr}\, MM^T = M_{ij}M_{ij} = \epsilon_{ijk} V_k \epsilon_{ijl} V_l =$$
$$= 2\vec{V}^2$$

Hence: $\frac{1}{2}\mathrm{tr}\, MM^T = \vec{V}^2$

And in general, if $M_{ij} = \epsilon_{ijk} V_k$; $N_{ij} = \epsilon_{ijk} w_k$

$$\frac{1}{2}\mathrm{tr}\, MN^T = \vec{V}\cdot\vec{w}$$