- Gradient flow:

◦ HW comment
◦ Linear independence

1

$$\frac{dx^i}{dt} = \frac{\partial f}{\partial x^i} \quad \rightarrow \quad \frac{dx^i}{dt} = \underline{\underline{g^{ij}}}\frac{\partial f}{\partial x^j}$$

↑ presence of metric

(example - honey descending the hills)

$\|V\|$ - norm (length of the vector)

$\langle \vec{v}, \vec{w} \rangle$

$\langle \vec{v}, \vec{v} \rangle = \|V\|^2$

$$\underbrace{\langle \vec{v}+\vec{w}, \vec{v}+\vec{w} \rangle}_{\|\vec{v}+\vec{w}\|^2} = \|V\|^2 + \|W\|^2 + 2\langle \vec{v}, \vec{w} \rangle$$

If distances are known → scalar product is well-defined.

$$\vec{V} = v^i \vec{e}_i$$

↑ basis vectors

$$g_{ij} \equiv \langle \vec{e}_i, \vec{e}_j \rangle \; ; \quad g^{ij} \equiv (g^{-1})^{ij}$$

- Hamiltonian flow
  Symplectic flow

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q_i}$$

- covariance is OK

Generalisation:

q, p are coordinates in 2d-space [advantage of Hamiltonian point of view]

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i} \; ; \quad \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}$$

$$\{q, p\} \rightarrow x^\alpha$$

$$\dot{x}^\alpha = \omega^{\alpha\beta}\frac{\partial H}{\partial x^\beta} \; ; \quad \omega^{\alpha\beta} = (\omega^{-1})^{\alpha\beta}$$

ω - symplectic form

Ask to figure themselves "metric" then puzzle them with antisymmetry

~~$\omega =$~~ $n=1$ $\{q, p\}$

$$\omega^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$ - antisymmetric.

$$\omega = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \qquad \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}$$

$\uparrow$

canonical basis
(Darbaux basis)

Compulsory program - <u>only canonical basis</u>

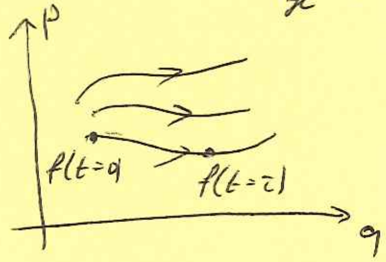
	generically	First study
Metric spaces	$g_{\alpha\beta}$ $\uparrow$ sym	Euclidean space $g_{\alpha\beta} = \delta_{\alpha\beta}$
Symplectic manifold	symplectic form $\omega_{\alpha\beta}$ $\uparrow$ antisym.	ω - <u>canonical one</u>

On symplectic manifold

any function defines a vector flow
e.g $\mathcal{H}$

$$V^{\alpha}_{\mathcal{H}} = \omega^{\alpha\beta} \frac{\partial \mathcal{H}}{\partial x^{\beta}}$$

$\cancel{f}$ $\quad f(p,q)$



$$\dot{x}^{\alpha} = V^{\alpha}_{\mathcal{H}}$$

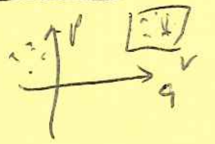
$$\frac{df(p,q)}{dt} = \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p^i}\dot{p}^i = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial \mathcal{H}}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial \mathcal{H}}{\partial q^i} = \{f, \mathcal{H}\}.$$

Poisson bracket

$$\frac{df(x)}{dt} = \frac{\partial f}{\partial x^{\alpha}}\dot{x}^{\alpha} = \frac{\partial f}{\partial x^{\alpha}}\omega^{\alpha\beta}\frac{\partial \mathcal{H}}{\partial x^{\beta}} = \{f, \mathcal{H}\}$$

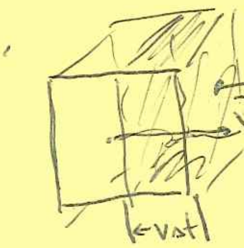
- Symmetry of the Poisson bracket.

Continuity equation



$$\rho = \frac{dN}{dV}$$

- story about bottle

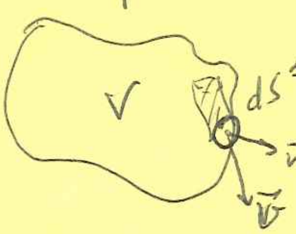


$$\rho(x) = \lim_{V \to 0} \frac{N}{V}$$

$\int \rho \, dV =$ total amount of particles

How many particles will leave the box in time Δt?

$$\Delta N = -V (v \Delta t \cdot S') \rho$$



$$\Delta N = -\rho (\vec{v} \cdot \vec{n}) \Delta t \cdot dS$$

$\Delta t \to 0$:

$$\frac{dN_V}{dt} = -\iint \rho (\vec{v}\vec{n}) \, dS \overset{\text{Stoke's theorem}}{=}$$

$$= -\iiint \operatorname{div}(\rho \vec{v}) \, dV$$

$$\operatorname{div} \vec{v} = \sum_i \frac{\partial v^i}{\partial x^i}$$

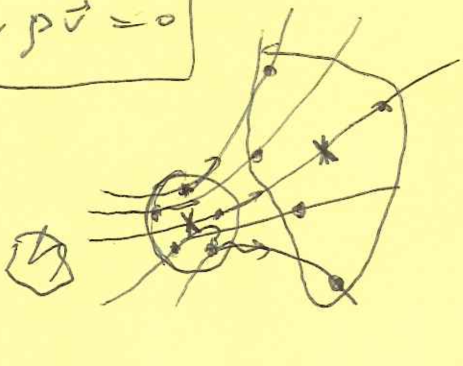
$$\frac{1}{V} \frac{dN_V}{dt} + \frac{1}{V} \iiint_V \operatorname{div}(\rho \vec{v}) \, dV = 0$$

$V \to 0$

$$\boxed{\frac{\partial \rho(x,t)}{\partial t} + \operatorname{div} \rho \vec{v} = 0}$$

~~made~~ a mistake $\frac{d\rho}{dt}$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x} \dot{x} + \frac{\partial \rho}{\partial t}$$



Liouville's theorem

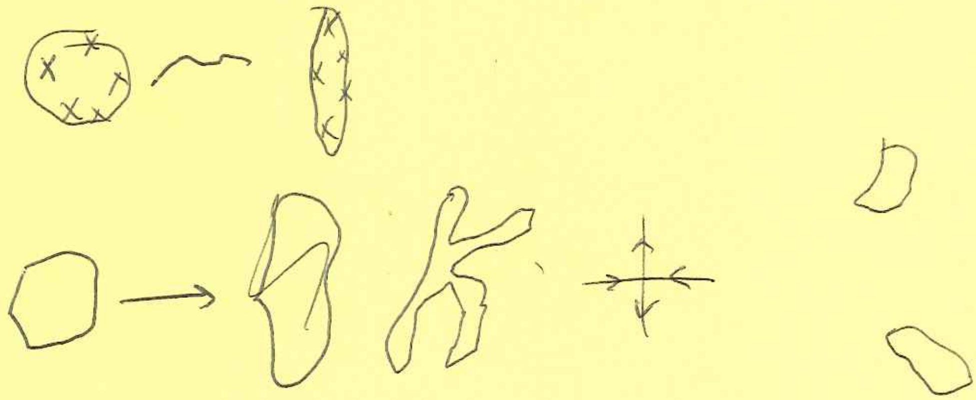
$$\cancel{\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{v}) = \frac{\partial \rho}{\partial t} + (\partial_i v^i)\, v^i \frac{\partial}{\partial x^\alpha}(\rho v^\alpha) = \frac{\partial\rho}{\partial t} +}$$

$$\dot{x}^\alpha = v^\alpha = \omega^{\alpha\beta}\frac{\partial H}{\partial x^\beta}$$

$$\frac{d\rho(x,t)}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x^\alpha}\dot{x}^\alpha = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x^\alpha}v^\alpha = \underbrace{\frac{\partial}{\partial t} + \frac{\partial}{\partial x^\alpha}(\rho v^\alpha)}_{0} - \rho\frac{\partial v^\alpha}{\partial x^\alpha} =$$

$$= -\rho\frac{\partial}{\partial x^\alpha}\left(\omega^{\alpha\beta}\frac{\partial H}{\partial x^\beta}\right) = -\rho\,\omega^{\alpha\beta}\frac{\partial^2 H}{\partial x^\alpha \partial x^\beta} = 0$$

ρ is an integral of motion



(mistake)

<u>Liouville's theorem</u>: Hamiltonian flow preserves ~~volume~~ phase volume ~~of the system~~

(weak, but most common statement)

<u>stronger statement</u> Hamiltonian flow preserves the symplectic structure

wrong: $\frac{d}{dt}\omega_{ij}(x) = \frac{\partial \omega_{ij}}{\partial x^\alpha}\dot{x}^\alpha = 0$