MA2326: ODE ? /

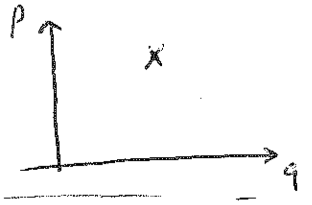
Phase portrait of a Hamiltonian system

Posts

$$\mathcal{H}(\underbrace{q_1, q_2, \dots, q_m}_{\text{generalised coord}}; \underbrace{p_1, p_2, \dots, p_m}_{\text{generalised momenta}})$$

$2m$-dimensional space of p's and q's is called <u>phase space</u>.

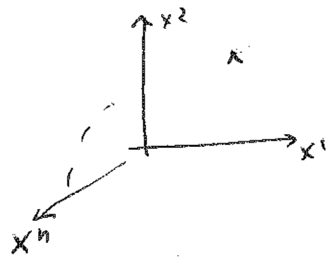
$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \; ; \; \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$



System of ODE's

n-dim. space $\{x^1, \dots, x^n\}$

$$\frac{dx^i}{dt} = v^i(x)$$

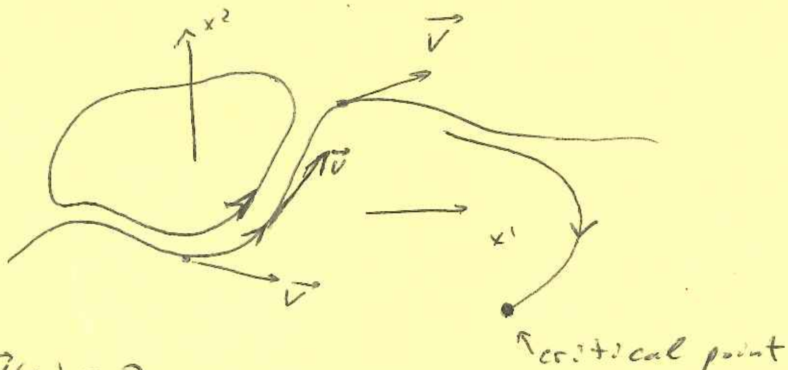


$$x^i(t+\Delta t) = x^i(t) + v^i(x(t))\,\Delta t + O((\Delta t)^2)$$

[Theorems about existence of solut, uniq etc.]

(MA2326: ODE)

Last time $\quad \frac{dx^i}{dt} = V^i(x)$ $\qquad$ ① FET



critical point x_0: $\vec{V}(x_0) = 0$

? what is the behaviour near critical point

$x^i = x_0^i + y^i$; $\vec{y}$ is small

$\underline{\dot{y}^i = A^i{}_j\, y^j}$; $A^i{}_j = \frac{\partial V^i}{\partial x^j}\Big|_{x=x_0}$

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②

$y^i(t) = c^i e^{\lambda t}$ $\qquad$ 2-trial solution

$\lambda c^i \cancel{e^{\lambda t}} = A^i{}_j c^j \cancel{e^{\lambda t}}$; $\hat{A}\cdot\vec{c} = \lambda\vec{c}$

characteristic equation

$\boxed{n \text{ solutions.}}$ $\{\lambda_\alpha, \vec{e}_{(\alpha)}\}$

General

$$y^i = \sum_{\alpha=1}^{n} B_\alpha e^i_{(\alpha)} e^{\lambda_\alpha t}$$

B_α ↑ constants of integration

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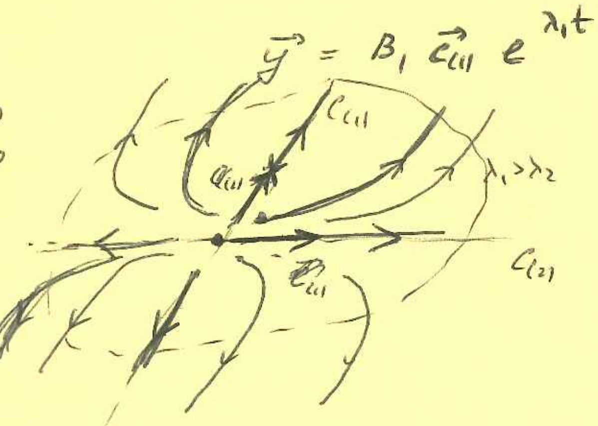
③

Consider 2-dimensional case:

$$\vec{y} = B_1 \vec{e}_{(1)} e^{\lambda_1 t} + B_2 \vec{e}_{(2)} e^{\lambda_2 t}$$

$\lambda_1 > 0$
$\lambda_2 > 0$



$\lambda_1 < 0$
$\lambda_2 < 0$
reverse all arrow

unstable Node

stable Node
$\underline{\infty \text{ time}}$ to reach critical point.

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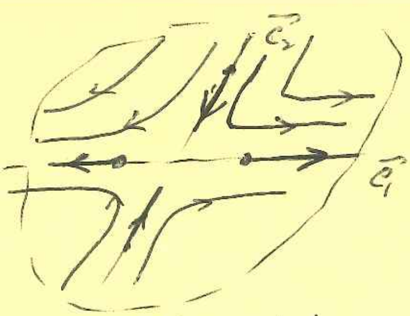
$\lambda_1 > 0$
$\lambda_2 < 0$

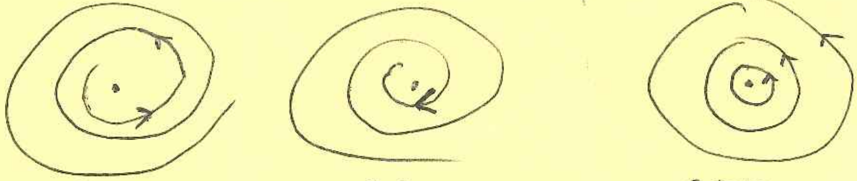


saddle point

$\lambda_1 = a + ib$
$\lambda_2 = a - ib$

$$\vec{y} = Re\left(B \vec{c} e^{(a+ib)t} \right)$$

↑ complex



$a > 0$ $a < 0$ $a = 0$

focal points | center

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② Formal solution [can skip]

$$\vec{y} = e^{\hat{A}t} \cdot \vec{y}_0 \quad ; \quad f(\hat{A}) = f(0) + \hat{A} f'(0) + \frac{1}{2} \hat{A}^2 f''(0) + \ldots =$$

$$\hat{A} = U \Lambda U^{-1} \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

↑ diagonal

$$\hat{A}^n = U \Lambda^n U^{-1}$$

$$f(\hat{A}) = U f(\Lambda) U^{-1}$$

$$e^{\hat{A}t} = U e^{\Lambda t} U^{-1} = U \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} U^{-1}$$

<u>Question:</u>

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