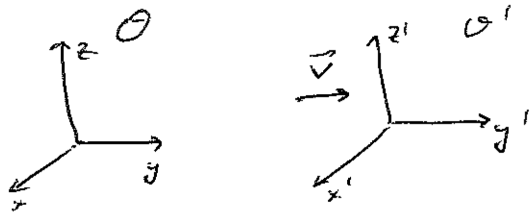
<u>Previous lecture:</u>



Change indices up ↔ down

<u>Lorentz boost</u>

$$\vec{x}' = \gamma(\vec{x} - \vec{\beta}(ct)) \quad ; \quad \vec{\beta} = \frac{\vec{V}}{c}$$

$$ct' = \gamma(ct - \vec{\beta}\vec{x}) \qquad \gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}}$$

Usually we consider only boost in x-direction:

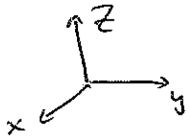
$$\vec{V} = \{V, 0, 0\}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$y' = y$$
$$z' = z$$

Last time we discussed about how quantities change. Today we discuss the quantities <u>that does not change</u>

<u>Analogy:</u> Consider 3-d space:



- Arbitrary linear transformation:

$$x_i' = G_i{}^j x_j \quad ; \quad G - 3\times 3 \text{ matrix}$$
$$\det G \neq 1$$

[~~analogy of~~ changing reference frame]

Subclass of transformations:

Orthogonal transformations
- Transformations that preserve distances ⟷ scalar product.

$$\vec{u}\cdot\vec{w} = \underbrace{\delta^{ij}}_{\text{Euclidean metric}} u_i w_j = u_1 w_1 + u_2 w_2 + u_3 w_3$$

$$\vec{u}'\vec{w}' = \delta^{ij} u_i' w_j'$$

$$\delta^{ij} u_i' w_j'$$

$\|\vec{u}\|$ - if you know norm, you can define scalar product

$$\langle \vec{u}, \vec{w} \rangle = \tfrac{1}{2}(\langle \vec{u}+\vec{w}, \vec{u}+\vec{w}\rangle - \langle \vec{u}, \vec{u}\rangle - \langle \vec{w}, \vec{w}\rangle) =$$
$$= \tfrac{1}{2}\,\|\vec{u}+\vec{w}\|^2 - \|\vec{u}\|^2 - \|\vec{w}\|^2$$

$$\vec{u}'\vec{w}' = \delta^{ij} u_i' w_j' = \delta^{ij} G_i^{\ i'} G_j^{\ j'} \underline{u_{i'} w_{j'}} = \delta^{i'j'} \underline{u_{i'} w_{j'}}$$

for arbitrary

$$\boxed{\delta^{ij} G_i^{\ i'} G_j^{\ j'} = \delta^{i'j'}}$$

$$GG^T = \mathbb{1}$$

usualy denotet by O: (G – for general)
(O – for orthogonal)

(*) $$\boxed{OO^T = \mathbb{1}}$$

<u>Property</u> if O_1, O_2 are orthogonal then O_1O_2 is orthogonal also.

$$(O_1O_2)(O_1O_2)^T = O_1 \underbrace{O_2 O_2^T}_{} O_1^T = O_1 O_1^T = \mathbb{1}.$$

=> the orthogonal transformations form a group : $O(3)$

[$O(n)$ for n-dimensional case]

<u>Geometric interpretation</u>:

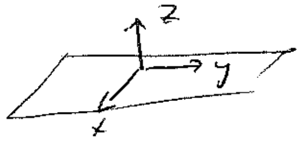
$$(*) \Rightarrow 1 = \det(OO^T) = (\det O)^2 \Rightarrow \det O = \pm 1$$

$\det O = 1 \Rightarrow$ rotations

<u>$SO(3)$</u>

$\det O = -1$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$



reflection

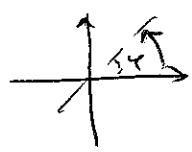
$$O = \underbrace{\left(O \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}\right)}_{\substack{\det\# = 1 \\ \text{(rotation)}}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

× reflection

$$O(3) = SO(3) \times \mathbb{Z}_2$$

<u>2d example</u>: (of a rotation)

$$O = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$



Orthogonal transformations preserve $\underline{\Delta x^2 + \Delta y^2 + \Delta z^2} = \Delta L^2$

(length in Euclidean space) between two points

Define Minkowski space: 4 dim space with "length" between two events
$\underline{1+3 \text{ dim space}}$ $\{ct, x, y, z\}$

$$\Delta s^2 = (c\Delta t)^2 - \Delta \vec{x}^2$$

Options: $\Delta s^2 > 0$
$\Delta s^2 = 0$: ~~Length~~ $\to$ Interval.
$\Delta s^2 < 0$

<u>Lorentz group:</u> Group of all linear transformations that preserve interval:

$$\vec{u}\,\vec{w} = \underbrace{\eta^{ij}}_{\substack{\text{Minkowsky} \\ \text{metric}}} u_i w_j = u_0 w_0 - u_1 w_1 - u_2 w_2 - u_3 w_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\vec{u}'\vec{w}' = \eta^{ij} u_i' w_j' = \eta^{ij} u_i w_j \Rightarrow$$

$\Rightarrow$ (full analogy with $O(3)$ case)

$$\eta^{ij} L_i^{\ i'} L_j^{\ j'} = \eta^{i'j'}$$

$$\boxed{L \eta L^T = \eta} \ ; \ L - \text{Lorentz transformation}$$

$$O(1,3)$$

$L_1 L_2$ - also Lorentz transform

Different types of transforms:

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & O & \\ 0 & & & \end{array}\right) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$\Delta s^2 = (c\Delta t)^2 - \Delta \vec{x}^2 \to$

$\to c(\Delta t)^2 - |O \Delta \vec{x}|^2$

3-d rotations + space reflection (P) are included.

$$\begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \text{Lorentz boosts (in x-direction)}$$

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Enough to consider 1+1 ~~case~~. Minkowsky space

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\uparrow}{\eta}} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}^T = \eta$$

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ -\beta\gamma & -\gamma \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & 0 \\ 0 & (\beta^2-1)\gamma^2 \end{pmatrix} =$$

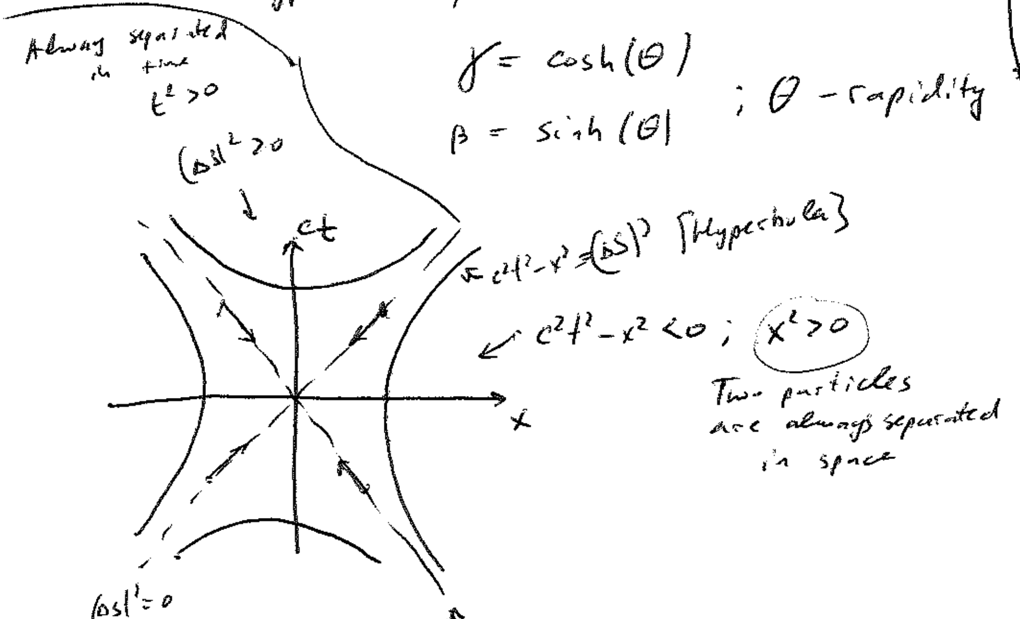
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

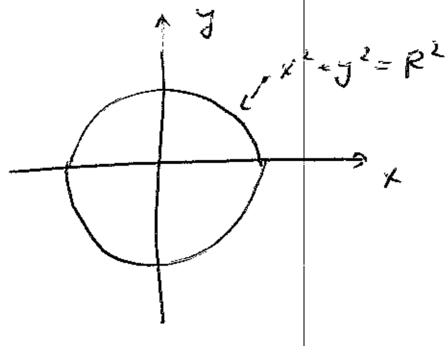
In reality we used $\gamma^2 - (\beta\gamma)^2 = 1$

analogy
$(\cos\varphi)^2 + (\sin\varphi)^2 = 1$

Hyperbolic parametrisation:

$\gamma = \cosh(\theta)$
$\beta = \sinh(\theta)$; θ - rapidity

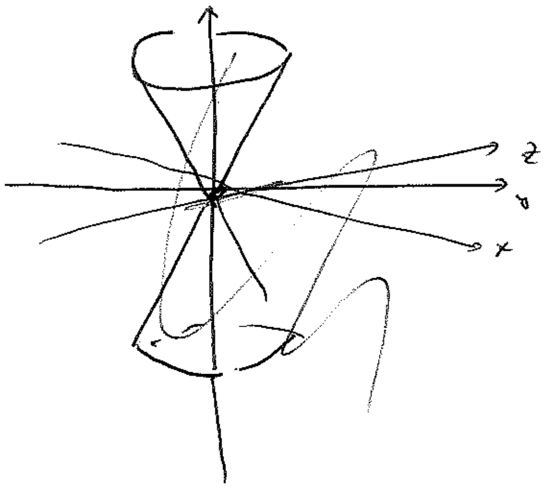


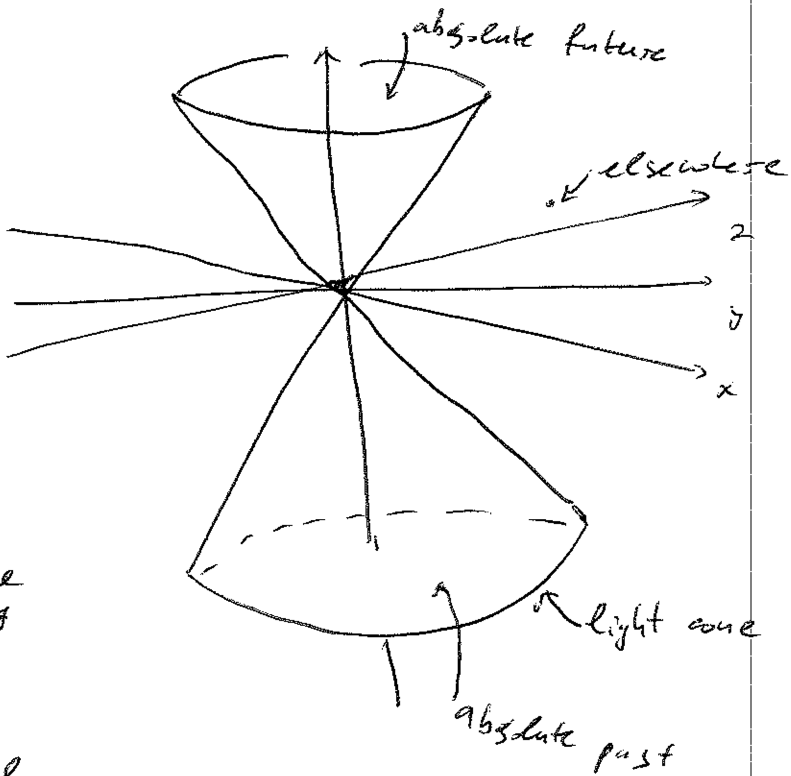


$(\Delta s)^2 = 0$ ← light-like interval

$c^2t^2 - x^2 = 0$

$\boxed{x = \pm ct}$ ← particle moving with the speed of light





$\Delta s^2 = 0$ - light like ~~trajectory~~ interval

$\Delta s^2 > 0$ - time-like interval

$\Delta s^2 < 0$ - space-like interval.

Lorentz transform does not mix these regions.

$$O(1,3) \simeq SO(1,3) \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$X \xleftrightarrow{\text{conjugate}} p$$

$$\cancel{t} \longleftarrow E$$

const. $(ct)^2 - \vec{x}^2$; $\quad \left(E/c\right)^2 - p^2 = \text{const} = m^2c^2$

$$E = \sqrt{(mc^2) + (cp)^2}$$

$p=0$ (particle at rest)

$$E = mc^2$$