① On covariant systems

Lecture

Maxwell equations, covariant form

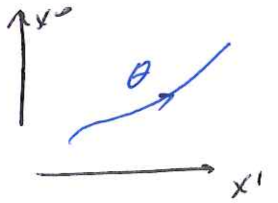
Not invariant statements:
- location in space, distance
- event at a given time, time spent

Insteded we should speak about:
- space-time event $(ct, \vec{x}) = (x^0, \vec{x})$ (covariant object, i.e with well-prescribed rule of transformation when changing frames)
- interval: $\Delta s^2 = c^2 \Delta t^2 - \Delta x^2$

Typical notation: $x^\mu = (x^0, \vec{x})$ a 4-vector

Greek $\mu, \nu, \ldots$ – denote components of 4-vectors

Latin $i, j, \ldots$ – denote spatial components



$x^\mu(\theta)$ – some trajectory

$$\frac{dx^\mu}{d\theta} = u^\mu \quad \text{4-velocity}$$

depends on the reference frame

If L – Lorentz transformation from one frame to another, i.e $\begin{pmatrix} ct' \\ \vec{x}' \end{pmatrix} = L \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$

Then $(u^\mu)' = L^\mu{}_\nu \, u^\nu$

u is nothing but a tangent vector or contra-variant vector.

(2) In Minkowsky space, the scalar product:

wrong !!! $\langle u,u\rangle \neq u^0u^0 + u^1u^1 + u^2u^2 + u^3u^3$ $\qquad \eta = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}$

<u>correct</u>: $\langle u,u\rangle = \eta_{\mu\nu} u^\mu u^\nu = u^0u^0 - u^1u^1 - u^2u^2 - u^3u^3$

Lorentz transformations were defined as those that preserve scalar product. Conclusion is that:

$$\underline{\eta_{\mu'\nu'} L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} = \eta_{\mu\nu}}$$

The value of the scalar product is the same in any Lorentz frame (it is invariant object)

If u^μ – contra-variant object, we can define co-variant object:

$\underline{u_\mu = \eta_{\mu\nu} u^\nu}$, obviously $\langle u,u\rangle = u_\mu u^\mu$

lowering

and in general $\langle u,w\rangle = u_\mu w^\mu = \eta_{\mu\nu} u^\mu w^\nu$

Define $\eta^{\mu\nu} = (\eta^{-1})^{\mu\nu}$. Clearly, $\eta^{-1} = \eta$

Then we can now raise indices also:

$$u^\mu = \eta^{\mu\nu} u_\nu.$$

Summary:

u^μ – contra-variant. If $u^\mu = (u^0, \bar{u})$

u_μ – co-variant Then $u_\mu = (u^0, -\bar{u})$

Transformation rule:

$$(u^\mu)' = L^\mu{}_\nu u^\nu$$
$$(u_\mu)' = u_\mu (L^{-1})^\mu{}_\nu$$

(check that $u^\mu u_\mu$ is invariant)

(3) We can also study tensors:

$T_{\mu\nu}$ – covariant 2-tensor

transformation rule: $T'_{\mu'\nu'} = T_{\mu\nu} (L^{-1})^{\mu}{}_{\mu'} (L^{-1})^{\nu}{}_{\nu'}$

T_{μ}^{ν} – (1,1) tensor

transformation rule: $T'^{\nu'}_{\mu'} = L^{\nu'}{}_{\nu}\, T^{\nu}_{\mu} (L^{-1})^{\mu}{}_{\mu'}$

etc.

$\eta_{\mu\nu}$ can be thought as a covariant 2-tensor.

It happens so that $\eta'_{\mu'\nu'} = \eta_{\mu\nu}$ (that is how Lorentz transformation is defined)

But if we consider some generic $G(x)$ ← G is for general
depends on coordinate

then $$\eta'_{\mu'\nu'} = \eta_{\mu\nu} (G^{-1})^{\mu}{}_{\mu'} (G^{-1})^{\nu}{}_{\nu'}$$

$\eta' \neq \eta$. We usually denote ~~g~~ $\eta' \equiv \boxed{g}$

$g_{\mu\nu}(x)$ is called space-time metric.

<u>Einstein</u>, <u>general relativity</u>: Gravity deforms space-time metric such that it cannot become η globally in any reference frame.

<u>Example</u>: Metric of a massive spherical object:

Schwarzshild black hole
~~rs~~ All mass is inside r_s

$$r_s = \frac{2GM}{c^2} \begin{pmatrix} 1-\frac{r_s}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

in coordinate system (ct, r, θ, φ)

(4) All physical laws should be in covariant form.

If $A = B$, then A & B should transform in the same way with respect to Lorentz transformations.

All laws of physics should written in the forms of equalities between Lorentz scalars, vectors, tensors.

We will discuss El.-Mag.

<u>Maxwell equations in Gaussian units:</u>

Gauss law: $\vec{\nabla} \cdot \vec{E} = 4\pi \rho$, ρ - charge density

$\vec{\nabla} \cdot \vec{B} = 0$ (no monopoles)

Faraday law: $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

Ampère's law: $\vec{\nabla} \times \vec{B} = \frac{1}{c} \left(4\pi \vec{J} + \underbrace{\frac{\partial \vec{E}}{\partial t}}_{\text{Lorentz correction}} \right)$, $\vec{J}$ - charge current.

⑤ Light as solution of Maxwell equations:

Maxwell in the absence of matter:

$$\vec\nabla \vec E = 0 \qquad \vec\nabla \times \vec E = -\frac{1}{c}\frac{\partial \vec B}{\partial t}$$

$$\vec\nabla \vec B = 0 \qquad \vec\nabla \times \vec B = +\frac{1}{c}\frac{\partial \vec E}{\partial t}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec E}{\partial t^2} = \vec\nabla \times \left(\frac{1}{c}\frac{\partial \vec B}{\partial t}\right) = -\vec\nabla \times (\vec\nabla \times \vec E) = -\vec\nabla \cdot \underbrace{(\vec\nabla \vec E)}_{=0} + (\vec\nabla^2)\vec E$$

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 \vec E}{\partial t^2} - (\vec\nabla^2)\vec E = 0 \\ \dfrac{1}{c^2}\dfrac{\partial^2 \vec B}{\partial t^2} - (\vec\nabla^2)\vec B = 0 \end{cases}$$

in the same way

a Solution: plane wave $\vec E \propto e^{i(\omega t - \vec k \vec x)}$. Dispersion relation:

$$\omega^2 = c^2 \vec k^2$$

want real solution: $\vec E = \vec E_0 \cos(\omega t - \vec k \vec x)$

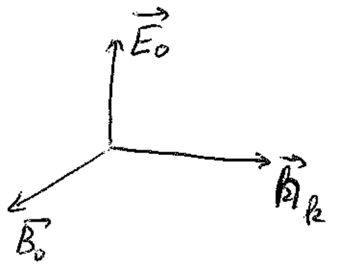
$$\vec B = \vec B_0 \cos(\omega t - \vec k \vec x)$$

recall that $\vec\nabla \vec E = 0$: $0 = \vec\nabla \vec E = \underbrace{(\vec k \cdot \vec E)}_{=0} \sin(\omega t - \vec k \vec x)$

hence $\vec k \perp \vec E$

also, from $\vec\nabla \vec B = 0$: $\vec k \perp \vec B$

now: $\vec\nabla \times \vec B = \frac{1}{c}\frac{\partial \vec E}{\partial t} \Rightarrow (\vec k \times \vec B_0)\sin(\omega t - \vec k \vec x) = -\frac{\omega}{c}\vec E_0 \sin(\omega t - \vec k \vec x)$

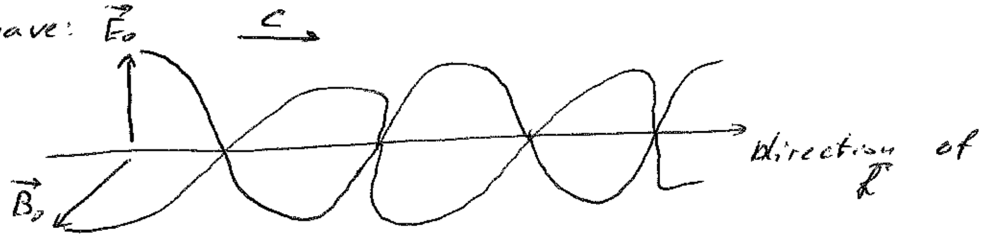


$$\vec k = |k| \cdot \vec n_k \ ; \quad |k| = \frac{\omega}{c}$$

$$(\vec n_k \times \vec B_0) = \vec E_0$$

Traveling wave:

[Light!]



⑥ Goal: rewrite Maxwell equations in covariant form:

$\vec\nabla \vec B = 0$

$\vec\nabla \times \vec E = -\frac{1}{c}\frac{\partial \vec B}{\partial t}$

If $\vec\nabla \cdot \vec B = 0$ then $\vec B = \vec\nabla \times \vec A$, $\vec A$ - vector potential

this statement is analog of

if $\partial_i a_j = \partial_j a_i$ then $a_j = \partial_j V$ [General statement: Poincare lemma]

More explicitly:

skip during lecture

$B_i = -\frac{1}{2}\epsilon_{ijk} F_{jk}$; $F_{ij} = \begin{pmatrix} 0 & -B_z & B_y \\ +B_z & 0 & -B_x \\ -B_y & +B_x & 0 \end{pmatrix}$

$\vec\nabla \cdot \vec B = 0 \Rightarrow \epsilon_{ijk}\partial_i F_{jk} = 0$

$\partial_i B_i = 0$

$\vec B = \vec\nabla \times \vec A \Rightarrow$

$B_i = \epsilon_{ijk}\partial_j A_k$

$\Rightarrow F_{ij} = -(\partial_i A_j - \partial_j A_i)$

$\epsilon_{ijk}\partial_i \cdot (-(\partial_j A_k - \partial_k A_j)) =$

$= -\epsilon_{ijk}\partial_i\partial_j A_k + \underbrace{\underset{\substack{\| \\ -\epsilon_{ikj}}}{\epsilon_{ijk}}\partial_i\partial_k A_j}_{\substack{\| \\ -\epsilon_{ijk}\partial_i\partial_j A_k}} = -2\underbrace{\epsilon_{ijk}}_{\text{asym}}\overbrace{\partial_i\partial_j}^{\text{sym}} A_k = 0$

$0 = \vec\nabla \times \vec E + \frac{1}{c}\frac{\partial \vec B}{\partial t} = \vec\nabla \times \vec E + \vec\nabla \times \left(\frac{1}{c}\frac{\partial \vec A}{\partial t}\right) = \vec\nabla \times \left(\vec E + \frac{1}{c}\frac{\partial \vec A}{\partial t}\right)$

$\vec E + \frac{1}{c}\frac{\partial \vec A}{\partial t} = -\vec\nabla\varphi$, φ - scalar potential

introduce 4-vector: $A = (A^0, \vec A)$; $A^0 = \varphi$

tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\cancel{\partial_\mu F_{\mu\nu}}$

Sign convention for derivative:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t} , \vec{\nabla} \right)$$

hence $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t} , -\vec{\nabla} \right)$

For vectors it was the opposite:

$$A^\mu = (A^0 , \vec{A})$$
$$A_\mu = (A^0 , -\vec{A})$$

E.g in

$$\vec{\nabla}\cdot\vec{E} = \underbrace{-\,\partial^i}_{\text{because of sign convention for } \partial} F_{0i}$$

(7) $F_{ij} = -(\partial_i A_j - \partial_j A_i) = -\epsilon_{ijk} B_k$

$$F_{0i} = \frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i} = -\frac{\partial (\vec{A})^i}{c\,\partial t} - \frac{\partial \varphi}{\partial x^i} = (\vec{E})^i$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\partial_{\{\mu} F_{\nu\lambda\}} = 0 \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

: summary for $\nabla \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

[check this !]

Equations with matter:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{J}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = -\partial^i F_{0i} = \underbrace{-\partial^0 F_{00}}_{=0} - \partial^i F_{0i} = -\partial^\mu F_{0\mu} = \partial^\mu F_{\mu 0} = \partial_\mu F^{\mu 0}$$

$$\underline{\partial_\mu F^{\mu 0} = 4\pi\rho}$$

$$(\vec{\nabla} \times \vec{B})_i = \epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j \left(-\tfrac{1}{2}\epsilon_{klm} F_{lm}\right) =$$

$$= -\tfrac{1}{2}\epsilon_{ijk}\epsilon_{klm}\partial_j F_{lm} = \partial_j F_{ji} = \partial_\mu F^{\mu i} - \partial_0 F^{0i} =$$

$$= \partial_\mu F^{\mu i} + \partial_0 F_{0i} = \partial_\mu F^{\mu i} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{E})_i$$

$$\underline{\partial_\mu F^{\mu i} = \frac{4\pi}{c}(\vec{J})^i}$$

⑧ Introduce 4-vector: $j = \left(4\pi\rho, \frac{4\pi \vec{J}}{c}\right)$ (4-current)

Then Maxwell equation will read:

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$$

$$\underbrace{\partial_\nu \partial_\mu}_{\text{sym}} F^{\overbrace{\mu\nu}^{\text{asym}}} = 0 \quad \Rightarrow \quad \underline{\partial_\nu j^\nu = 0}$$

Physical meaning $\quad 4\pi\left(\frac{\partial \rho}{\partial x^0} + \frac{1}{c}\frac{\partial J^i}{\partial x^i}\right) = 0$

$$\frac{4\pi}{c}\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J}\right) = 0$$

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J} = 0} \quad \leftarrow \text{continuity equation,}$$

Summary: Maxwell equations in covariant form:

$$\boxed{\begin{array}{l} \partial_{[\mu} F_{\nu\lambda]} = 0 \\ \partial_\mu F^{\mu\nu} = j^\nu \end{array}} \quad \begin{array}{l} \Leftrightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \Rightarrow \quad \partial_\nu j^\nu = 0 \quad \text{(conservation of charge)} \end{array}$$

~~current~~