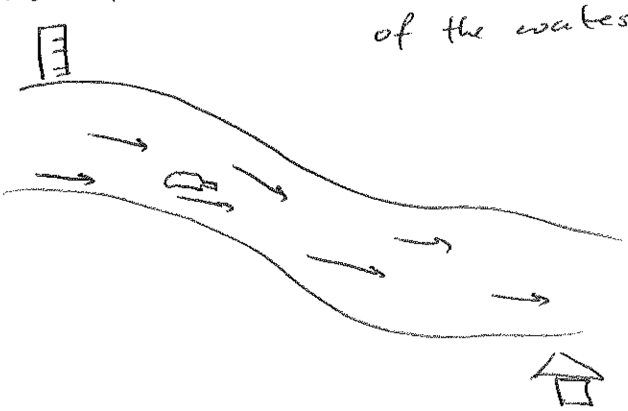
Lecture 1

continuity equation

Story about bottle

bottle ~~flo~~ follows the [vector] flow of the river. That is how speed of the water flow is measured.



∘ vector flow can be thought as movement of particles:

- hydrodynamics
- aerodynamics
- plasma.

Summary of what we learned

$$\mathcal{L}(q,\dot q) \xrightarrow[\text{transform}]{\text{Legendre}} \mathcal{H}(q,p)\,; \quad \dot q = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot p = -\frac{\partial \mathcal{H}}{\partial q} \qquad \underline{X = (q,p)}$$

symplectic structure

$$\rightarrow \quad \dot x^\alpha = \omega^{\alpha\beta} \frac{\partial \mathcal{H}}{\partial x^\beta} \longrightarrow \text{Liouville theorem}$$

← today's goal

symplectic [Hamiltonian] flow

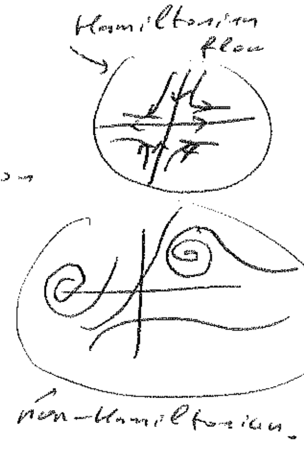
$\cap$

generic vector flow equations: $\dot x^\alpha = V^\alpha(x) \longrightarrow$ Continuity equation

$\cup$

$$\dot x^\alpha = g^{\alpha\beta} \frac{\partial V}{\partial x^\beta}$$ ← gradient [potential] flow

metric



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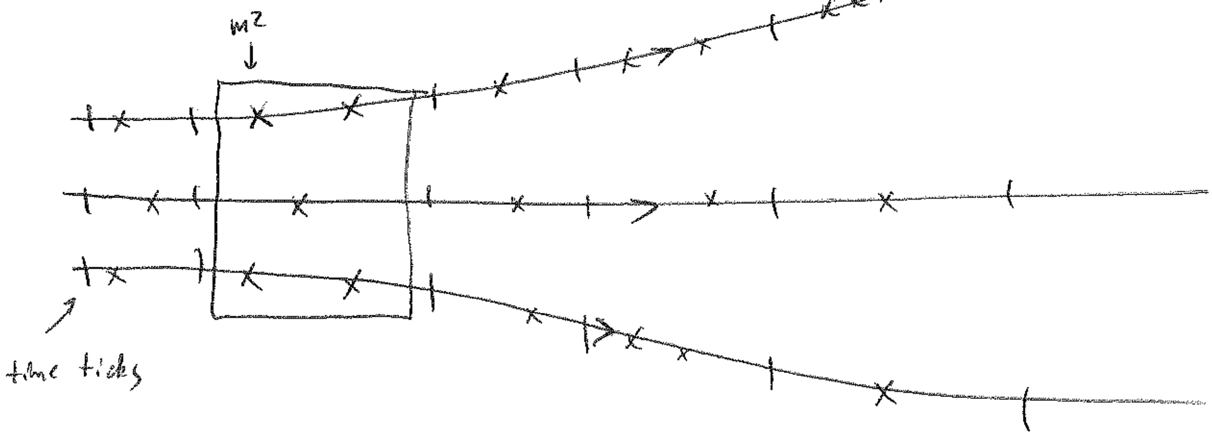
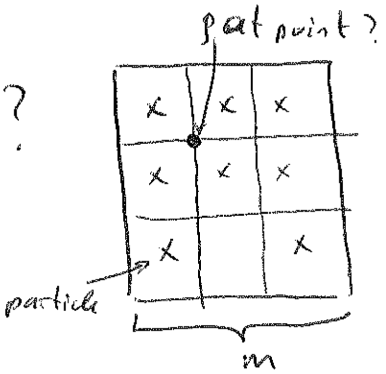
density: 5 particles/m^2



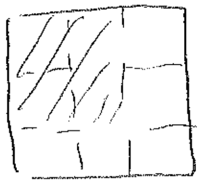
Particle density: $\rho = \frac{N}{V}$

$\lim_{V \to 0}$?

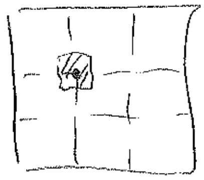


Approximations

$\rho = 8\ P/m^2$



$\rho = \frac{4\ P}{4/9\ m^2} = 9\ P/m^2$



$\rho = 0\ P/m^2$??

$V \to 0$ only in physical sense! It should be small compared to ~~the resolution of~~ our measurement precision, but still contain many particle.

If you want make $V \to 0$ mathematically, introduce δ-functions.

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cont. eq.

Difference between full derivative and partial derivative.

$$\frac{\partial \rho(x,y,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{\rho(x,y,t+\Delta t) - \rho(x,y,t)}{\Delta t}$$

measures change of ρ in a given point of space

$$\frac{d\rho(x,y,t)}{dt} = \lim_{\Delta t \to 0} \frac{\rho(x(t+\Delta t), y(t+\Delta t), t+\Delta t) - \rho(x,y,t)}{\Delta t}$$

place to measure ρ travels in time as well.

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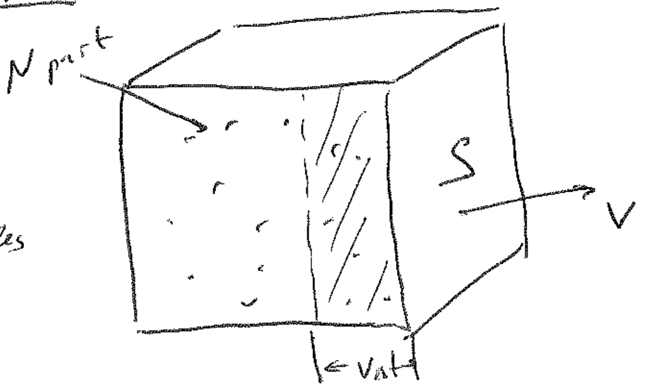
Continuity equation



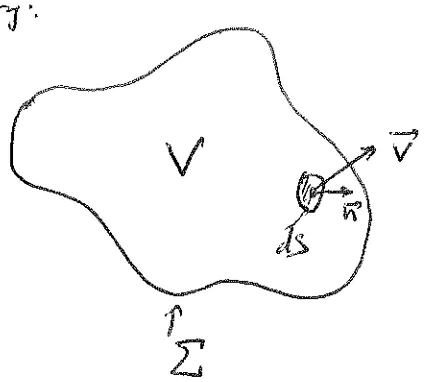
simple geometry
flow of particles
through the
border

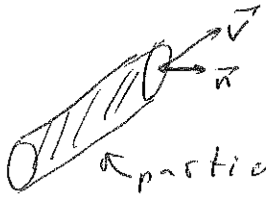
ΔN during time interval Δt?

$$\Delta N = -\rho(V\Delta t)S$$

$$\frac{\Delta N}{\Delta t} = -\rho V S \quad \leftarrow \text{rate of particle change}$$

general geometry:





particles in this
cylinder will
flow away

Volume of the cylinder:

$$(\vec{v}\vec{n}).ds \cdot \Delta t$$

# of particles there:

$$\rho(\vec{v}\vec{n})ds \cdot \Delta t$$

Stocke's theorem

$$\frac{dN}{dt} = -\iint_{\Sigma} \rho(\vec{v}\vec{n})\,dS \overset{\downarrow}{=} -\iiint_{V} div(\rho\vec{v})\,dV$$

$$\frac{d}{dt}\frac{N}{V} = -\frac{1}{V}\iiint_{V} div(\rho\vec{v})dV$$

$V \to 0$: $\frac{N}{V} \to \rho$, $\frac{d}{dt}\frac{N}{V} \to \frac{\partial\rho}{\partial t}$ (partial derivative because we keep volume fixed, not moving with time)

$$\frac{1}{V}\iiint_{V} div(\rho\vec{v})dV \to div(\rho\vec{v})$$

So, in $V \to 0$ we get:

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continuity equation

[Liouville's theorem]

$$(*) \quad \boxed{\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho \vec{v})} \quad \leftarrow \text{continuity equation}$$

$$\operatorname{div}(\rho \vec{v}) = \frac{\partial}{\partial x^i}(\rho \dot{x}^i) = \dot{x}^i \frac{\partial \rho}{\partial x^i} + \rho \frac{\partial \dot{x}^i}{\partial x^i} = \vec{v}\,\vec{\nabla}\rho + \rho \operatorname{div}\vec{v}$$

Hence, Another way to write continuity equation:

$$\frac{d\rho}{dt} = \underbrace{\frac{\partial \rho}{\partial x^i}\dot{x}^i}_{\vec{v}\,\vec{\nabla}\rho} + \underbrace{\frac{\partial \rho}{\partial t}}_{= -\vec{v}\,\vec{\nabla}\rho - \rho \operatorname{div}\vec{v}} = -\rho \operatorname{div}\vec{v}$$

$$(**) \quad \boxed{\frac{d\rho}{dt} = -\rho \operatorname{div}\vec{v}}$$

For Hamiltonian flow $\operatorname{div}\vec{v} = 0$:

$$\operatorname{div}\vec{v} = \frac{\partial}{\partial x^i} v^i = \frac{\partial}{\partial x^i}\left(\omega^{ij}\frac{\partial \mathcal{H}}{\partial x^j}\right) \underset{\omega^{ij} = \text{const}}{=} \overbrace{\omega^{ij}}^{\substack{\text{antisymmetric } i \leftrightarrow j}} \underbrace{\frac{\partial^2 \mathcal{H}}{\partial x^i \partial x^j}}_{\substack{\text{symmetric} \\ i \leftrightarrow j}} = 0$$

Hence, from $(**)$, for Hamiltonian flow:

$$\frac{d\rho}{dt} = 0$$

Liouville's theorem: Phase volume is conserved by Hamiltonian flow.

continuity equation

$\frac{d}{dt}\omega_{ij} = 0$ because ω is a constant matrix is a wrong conclusion!

ω_{ij} is a vector-type object. The value of ω_{ij} depends on a coordinate system we are working in.

Example: How $\vec{AB}$ changes by Ham. flow:

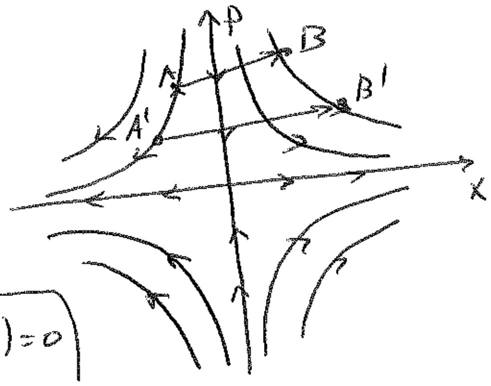
For $H = xp$:

$\dot{x} = x$

$\dot{p} = -p$

$\vec{v} = \begin{pmatrix} x \\ -p \end{pmatrix}$

$div\,\vec{v} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial p}(-p) = 0$



$\vec{AB}$ and $\vec{A'B'}$ are not the same!

$A \to A'$
$B \to B'$ ← not parallel trajectories.

Different-type objects:

		Example
"numbers"	scalar function	$f(p,q) = 3$
"direction"	tangent vector	$\vec{v}(p,q) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$
dual "direction"	differential form	$\alpha = 2dq + 3dp$
$\#/V$	density function	$\rho(p,q) = 5$

All examples seem to be constants. But:

$$\frac{df}{dt} = 0\ ;\quad \frac{d\vec{v}}{dt} \neq 0\,!\ ;\quad \frac{d\alpha}{dt} \neq 0\,!\ ;\quad \frac{d\rho}{dt} = -\rho\, div\,\vec{v} = 0$$

Only because the vector flow is Hamiltonian.

continuity equation [gener. Liouville's theorem].

How symplectic structure transforms when we change frame?

ω - symplectic structure in frame x $\qquad \underline{X} = \underline{X}(x)$

W - symplectic structure in frame $\underline{X}$

change of frame.

$$\{f,g\}_x \equiv \frac{\partial f(x)}{\partial x^\alpha} \omega^{\alpha\beta} \frac{\partial g(x)}{\partial x^\beta} = \frac{\partial f}{\partial \underline{X}^{\alpha'}} \underbrace{\frac{\partial \underline{X}^{\alpha'}}{\partial x^\alpha} \omega^{\alpha\beta} \frac{\partial \underline{X}^{\beta'}}{\partial x^\beta}} \frac{\partial g}{\partial \underline{X}^{\beta'}}$$

want that $\{\}$ is scalar $\to$ $\|$

$$\{f,g\}_{\underline{X}} \equiv \frac{\partial f}{\partial \underline{X}^\alpha} W^{\alpha\beta} \frac{\partial g}{\partial \underline{X}^\beta}$$

$$\boxed{W^{\alpha'\beta'} = \frac{\partial \underline{X}^{\alpha'}}{\partial x^\alpha} \omega^{\alpha\beta} \frac{\partial \underline{X}^{\beta'}}{\partial x^\beta}} \quad (*)$$

transformation rule.

$\dfrac{\partial \underline{X}}{\partial x}$ depends on $t \Rightarrow$

$$\Rightarrow \frac{dW^{\alpha\beta}}{dt} \neq 0 \quad \text{for generic vector flows.}$$

$(*)$: $$W^{\alpha\beta} = \frac{\partial \underline{X}^\alpha}{\partial x^{\alpha'}} \omega^{\alpha'\beta'} \frac{\partial \underline{X}^\beta}{\partial x^{\beta'}} = \{\underline{X}^\alpha, \underline{X}^\beta\}_x$$

poisson bracket in x-frame & $\underline{X}$ are considered as functions of x.

$$\left.\frac{dW^{\alpha\beta}}{dt}\right|_{t=0} = \{\dot{\underline{X}}^\alpha, \underline{X}^\beta\} + \{\underline{X}^\alpha, \dot{\underline{X}}^\beta\} =$$

$t=0$: for simplicity, can be relaxed

$$\underset{\substack{t=0 \\ \text{after differentiation}}}{\overset{\text{because}}{=}} \{v^\alpha(x), x^\beta\} + \{x^\alpha, v^\beta(x)\} =$$

$$= \frac{\partial v^\alpha}{\partial x^{\alpha'}} \omega^{\alpha'\beta'} \frac{\partial x^\beta}{\partial x^{\beta'}} + \frac{\partial x^\alpha}{\partial x^{\alpha'}} \omega^{\alpha'\beta'} \frac{\partial v^\beta}{\partial x^{\beta'}} = \frac{\partial v^\alpha}{\partial x^{\alpha'}} \omega^{\alpha'\beta} + \omega^{\alpha\beta'} \frac{\partial v^\beta}{\partial x^{\beta'}} =$$

$$= \omega^{\alpha'\beta} \omega^{\alpha\gamma} \frac{\partial^2 H}{\partial x^{\alpha'} \partial x^\gamma} + \omega^{\alpha\beta'} \omega^{\beta\gamma} \frac{\partial^2 H}{\partial x^{\beta'} \partial x^\gamma} = 0.$$

Conclusion $\frac{dW^{\alpha\beta}}{dt} = 0$

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continuity equation

Gen. Liouville's theorem

Generalised Liouvilles theorem: Hamiltonian flow preserves its symplectic structure.

Note: to construct vector field from Hamiltonian, we need a symplectic structure. It is exactly this symplectic structure which is preserved under the flow.

Hamiltonian flow at each moment of time can be considered as a coordinate transformation. $X = X(x)$

Since this preserves symplectic structure, EOM do not change:

By general covariance:

$$\frac{dx^\alpha}{dt} = \omega^{\alpha\beta} \frac{\partial \mathcal{H}}{\partial x^\beta} \underset{\text{coord.}}{\overset{\text{in new}}{\Longrightarrow}} \frac{dX^\alpha}{dt} = W^{\alpha\beta} \frac{\partial \mathcal{H}}{\partial X^\beta}$$

But $W = \omega$ by Liouville's theorem.

Note: $\omega_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the (q,p) basis

$\omega^{\alpha\beta} \equiv (\omega^{-1})^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the (q,p) basis