A bit smaller:

Landau

Hamiltonian mechanics

known: EOM in Lagrangian formulation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

← 2nd order diff eqn
# of eqns = s = # of q's (DOF)

Goal: rewrite EOM as 2s 1-st order diff eqns.

Generalised momenta: $p = \dfrac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}$ (1)

$$\dot{p} = \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q}$$ ← another one

Solve (1) for $\dot{q}$: $\dot{q} = \dot{q}(p,q)$ One 1-st order diff eqn

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$\dot{q} = v$; $\mathcal{L}(q,v)$

$\mathcal{L}(q,v)$: $\underset{*_1}{d\mathcal{L}} = \dfrac{\partial \mathcal{L}}{\partial q}dq + \dfrac{\partial \mathcal{L}}{\partial v}dv = \left| \begin{array}{c} p = \dfrac{\partial \mathcal{L}}{\partial v} \\ \Downarrow \\ v = v(q,p) \end{array} \right| =$

$$= \frac{\partial \mathcal{L}}{\partial q}dq + p\cdot\left(\frac{\partial v}{\partial q}dq + \frac{\partial v}{\partial p}dp\right) =$$

$$= \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial q}}_{\text{EOM: } \dot{p}} + p\frac{\partial v}{\partial q}\Big)dq + p\frac{\partial v}{\partial p}dp$$

$*_1$ should be aligned with $*_2$

$\mathcal{L}(q, v(q,p))$
$\parallel$
$\mathcal{L}[q,p]$

~~$\mathcal{L}(q, v(q,p))$~~:
$\mathcal{L}(q,p)$ $\underset{*_2}{d\mathcal{L}} = \dfrac{\partial \mathcal{L}}{\partial q}\Big|_p dq + \dfrac{\partial \mathcal{L}}{\partial p}dp$

*1
*2

1|4

- $\frac{\partial Z}{\partial v} dv \neq \frac{\partial Z}{\partial p} dp$? $\frac{\partial Z}{\partial v}\Big|_q \frac{\partial v}{\partial p} = \frac{\partial Z}{\partial p}\Big|_q$

Introduce color :

(write on top of previous)

$$dZ = \frac{\partial Z}{\partial q}\Big|_v dq + \frac{\partial Z}{\partial v}\Big|_q dv$$

color: $\quad \parallel \quad \neq \quad \neq$

$$dZ = \frac{\partial Z}{\partial q}\Big|_p dq + \frac{\partial Z}{\partial p}\Big|_q dp$$

} always two relations

$$\frac{\partial Z}{\partial q}\Big|_v \neq \frac{\partial Z}{\partial q}\Big|_p$$

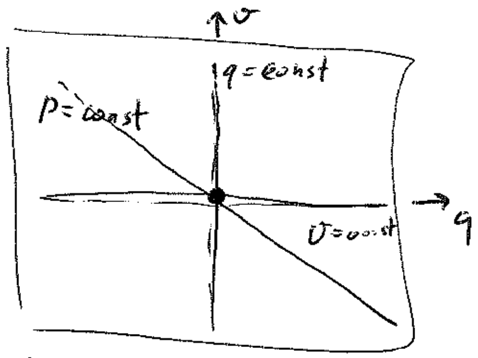
Example: $z = x + y$; $f(x, y) = x \cdot y$; $\frac{\partial f}{\partial x}\Big|_y = y$

$f(x, y(x,z)) = x \cdot (x+z)$; $\frac{\partial f}{\partial x}\Big|_z = 2x + z = x + y$

4|1

dv and dp are elements of V^* (2d vectors ?!)

$p = q + v$



$dv(\uparrow) = 1$; $dv(\rightarrow) = 0$

$dq(\uparrow) = 0$; $dq(\rightarrow) = 1 = dq(\searrow)$

$dp(\uparrow) = 1$; $dp(\rightarrow) = 1$

$dp(\searrow) = 0$

$dp = dq + dv$

〰 : v, q - basis

— : p, q - basis.

dv and dp are not collinear ~~vectors~~ ?

$$\dot p = \frac{\partial \mathcal{L}}{\partial q}\Big|_p - p\,\frac{\partial v}{\partial q}\Big|_p = -\frac{\partial}{\partial q}(pv - \mathcal{L})\Big|_q \qquad (11)$$

$$p\,\frac{\partial v}{\partial p}\Big|_q = \frac{\partial \mathcal{L}}{\partial p}\Big|_q \qquad \text{(we want } v = \dots \text{)}$$

$$\cancel{q}\; v = \frac{\partial}{\partial p}(pv)\Big|_q - p\,\frac{\partial v}{\partial p}\Big|_q = \frac{\partial}{\partial p}(pv - \mathcal{L})\Big|_q$$

When we make a change of variables $(q,\dot q) \to (q,p)$ we remark that there is a good combination to consider

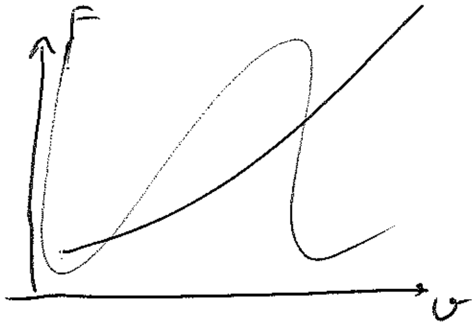
<u>Hamiltonian:</u> $\mathcal{H} = p\dot q - \mathcal{L}$

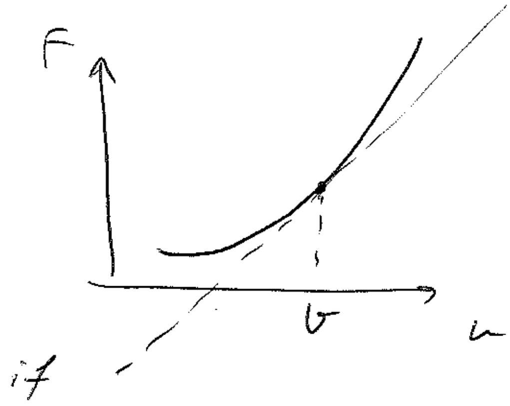
<u>Eqs:</u> $\dot p = -\dfrac{\partial \mathcal{H}}{\partial q}$; $\dot q = \dfrac{\partial \mathcal{H}}{\partial p}$

Legendre transform

$f(v) \rightsquigarrow g(p)$

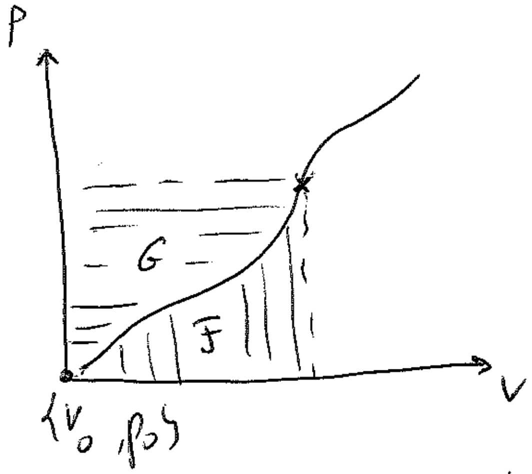
Consider function $F(v)$





slope $p = \frac{\partial F}{\partial v}$ at each point v

If $\frac{\partial^2 F}{\partial v^2} > 0$ (convex) then $\frac{\partial F}{\partial v}$ can be used instead of v



$$F = \int_{V_0}^{V} p\, dv$$

F is area under the curve

introduce G is area above the curve

$$\boxed{G + F = p \cdot V}$$

$G(p) = pV - F$ where $p = \frac{\partial F}{\partial V}$ is called the Legendre transform

Alternatively: $G(p) = \max_{V} (pV - F(V))$

$$\frac{\partial}{\partial V}(pV - F) = 0 \to p = \frac{\partial F}{\partial V}$$

max: $\frac{\partial^2 F}{\partial V^2} > 0$

11 $\mathcal{F}(\lambda, v)$

$$d\mathcal{F} = \frac{\partial F}{\partial \lambda}\Big|_v d\lambda + \underbrace{\frac{\partial F}{\partial V}\Big|_\lambda}_{= p} dv$$

$$\cancel{d(G}\; dG = d(\cancel{pv} - \mathcal{F} + pv) = -\frac{\partial F}{\partial \lambda}\Big|_v d\lambda + \cancel{p\,dv\; d(pv)} -$$

$$- p\,dv + d(pv) =$$

$$= -\frac{\partial F}{\partial \lambda}\Big|_v d\lambda + v\,dp$$

$$dG = \frac{\partial G}{\partial \lambda}\Big|_p d\lambda + \frac{\partial G}{\partial p}\Big|_\lambda dp$$

$$\frac{\partial G}{\partial p}\Big|_\lambda = v \;\; ; \longleftrightarrow \;\; \frac{\partial F}{\partial v}\Big|_\lambda = p$$

$$\frac{\partial G}{\partial p}\Big|_\lambda = v \longleftrightarrow \frac{\partial F}{\partial V}\Big|_\lambda = p$$

$$\boxed{\frac{\partial F}{\partial \lambda}\Big|_v = -\frac{\partial G}{\partial \lambda}\Big|_p}$$

14 $\mathcal{H}$: is ~~the~~ Legendre transform of $\mathcal{L}$

$\mathcal{H} \equiv G$

$\mathcal{L} \equiv \mathcal{F}$

$q \equiv \lambda$

$$\mathcal{H} = pv - \mathcal{L}(q, v) ; \quad p = \frac{\partial \mathcal{L}}{\partial v}$$

$$v \equiv \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \; , \qquad \underset{\substack{\uparrow \\ \text{EOM} \\ q}}{\dot{p}} = \frac{\partial \mathcal{L}}{\partial q}\Big|_v = -\frac{\partial \mathcal{H}}{\partial q}\Big|_p$$

Generically, p <u>is not</u> $p = m\dot{q}$

<u>Example 1</u>: $L = \frac{mv^2}{2}$; $H = \frac{p^2}{2m}$

<u>Example 2</u> $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2)$; $p_r = m\dot{r}$, $p_\varphi = mr^2\dot{\varphi}$ | $H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2}$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{r^3}$$

Announcement: • HW is online
• simpler derivation

Poisson brackets

Assumption: no explicit dependence on time (only through p and q)

$f(p,q)$

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} = \cancel{\{H,f\}}$$

$$\{f,g\} \equiv \sum_{i=1}^{s} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

$\cancel{\frac{df}{dt} = \{H,f\}}$

$\{H,p\} =$ $\quad \cancel{\{p,q\}} \quad \{p,q\} =$

$\cancel{\{H,q\}} =$

$\{p,p\} =$ $\qquad \{q,q\} =$ $\qquad \{p,q\} =$

$\{H,H\} =$

- $\{f,g\} = -\{g,f\} \rightarrow \{f,f\} = 0$
- $\{f, g_1 + g_2\} = \{f, g_1\} + \{f, g_2\}$

$\{f, g_1 g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$

$\{f,g\} = D_f \circ g \ ; \ D_f = \frac{\partial f}{\partial p}\cdot\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\cdot\frac{\partial}{\partial p}$

- $\{f, \{g,h\}\} = \{\{f,g\},h\} + \{g, \{f,h\}\}$

~~Def~~ I is called an integral of motion if $\frac{dI}{dt} = 0$

if $\cancel{\frac{\partial f}{\partial t}}$ $\frac{\partial I}{\partial t} = 0$ then $\frac{dI}{dt} = 0 \Leftrightarrow \{I, H\} = 0$

Poisson theorem: If I_1, I_2 are integrals of motion then $\{I_1, I_2\}$ is.