0. Vector space
1. Dual vector space
2. Differential:
   a) "Phys." small change of something [error ~~ff~~ in measurement] (OK because of)
      (Example?)
   b) $\underbrace{\text{Differential}}$ $\underbrace{\text{of a function}}$
      ↘ df
   c) Differential 1-form ; a means c
      b is called <u>exact differential</u>
3. Integration of differential 1-form

1h.

0. Vector space

$v, w \in V$: $v + w \in V$; $\lambda v \in V$;

$\lambda \in \mathbb{R}$: vector space over $\mathbb{R}$

$\lambda \in \mathbb{C}$: vector space over $\mathbb{C}$

[other fields possible
$\lambda \in K$: vector space over K]
(↑ some field)

space to say about dimension later

~~1.2 Dual vector~~

$\dim V$: max # of linearly independent vectors

2bls

[Wikipedia]

1. Dual space

V^* : space which is dual to V

Generically: V^* - space of functions from ~~on~~ V to $\mathbb{R}$. [space of your marks]
if you can get any mark

Typically: V^* also ~~preserve~~ respects structure of V

if V is manifold, V^* - smooth functions

~~V^* is~~

dual space V^* is defined as space of linear maps from V to $\mathbb{R}$.

$f \in V^*$ iff : $f : V \to \mathbb{R}$

$$\left.\begin{aligned} f(v+w) &= f(v) + f(w) \\ f(\lambda v) &= \lambda f(v) \end{aligned}\right\} \leftarrow \text{linearity}$$

3hd ~~[illegible]~~ $\boxed{1/4}$

V^* is also a vector space

$f, g \in V^* \to f+g \in V^*$

$\lambda \in \mathbb{R} \qquad \lambda f \in V^*$

$$(f+g)(\lambda v) = \lambda \cdot (f+g)(v) = \lambda(f(v)+g(v))$$

$$f(\lambda v) + g(\lambda v) = \lambda f(v) + \lambda g(v)$$

$$(f+g)(v+w) = (f+g)(v) + (f+g)(w)$$

$$(\lambda f)(\mu v) = \mu \cdot (\lambda f)(v)$$

$$(\lambda f)(v+w) = (\lambda f)(v) + (\lambda f)(w)$$

(2)

4hd ~~[illegible]~~ $\boxed{14}$ dim V^* - ? (remind that dim V is ...)

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basis in $V \cong \mathbb{R}^n$: $\vec{e}_1 = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$; $\vec{e}_2 = \begin{pmatrix}0\\1\\0\\ \vdots\end{pmatrix}$; ... ; $\vec{e}_n = \begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix}$

claim: a Basis in V^* is $\{\vec{e}^{\,i}\}$, $i \in \overline{1,n}$ defined

by $e^i(e_j) = \delta^i_j$

~~Kr. symbol~~

~~$f(v) = \sum v_i f$~~

linearity

~~$f(e_i) = \sum$~~

Any $\forall f \in V^*$ can be represented as $f = \sum_{i=1}^{n} f(\vec{e}_i)\,\vec{e}^{\,i}$; ~~$f(\vec{e}_j) = \sum_{i=1}^{n}$~~

dim $V^* = n$

$\boxed{14}$ $v \in V$

$u \in V^*$

$u(v) = \sum_{i=1}^{n} u_i v^i = u_i v^i$

Note: To define dual space we do not need metric ~~which raises and lowers indices~~

But if we have metric, we can identify V and V^*

$v_i, w_j \in V \to \langle v, w \rangle = v^i \underbrace{w^j g^{jl} g_{ij}}_{w_i}$

Metric raises and lowers indices

Erase all

2. Differential

a.) "Phys" : Differential is a small change of something in some process

[ok because ~~of error in measurement]~~

they know what ~~the~~ small means]

Example : measure blackboard in meters

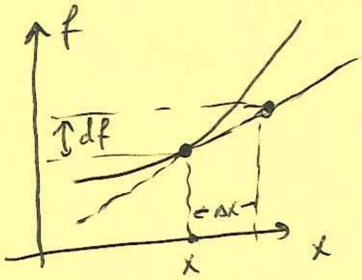
$$L(T) \; ; \quad \underbrace{\delta L = \frac{\partial L}{dT} \cdot \delta T}$$ - for all practical purposes

b) Differential of a function

$$df \; ; \qquad (df)(x, \Delta x) \equiv \frac{df}{dx} \cdot \Delta x$$

Space of displacements $\Delta x \simeq \mathbb{R}$

$$(df)(x) \in \mathbb{R}^*$$



$$dx = 1 \cdot \Delta x$$

$$df = \frac{\partial f}{dx} \cdot dx$$

Many variables:

$$\left[(df)(x_1^1, \dots, x_n^1)\right](\Delta x_1, \Delta x_2, \dots \Delta x_n) =$$

$$= \sum_{i=1}^{n} \frac{\partial f(\vec{x})}{\partial x^i} \Delta x_i$$

Δx_i - basis : dx^i - dual basis

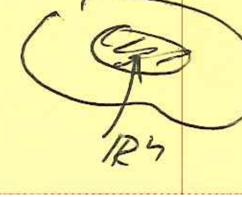
$(\vec{e}_i)$ $\qquad (\vec{e}^{\,i})$

$$\boxed{df = \sum_{i=1}^{n} \frac{\partial f(\vec{x})}{\partial x^i} dx^i}$$

Assumed that $x^1 .. x^n$ - coordinates of $\mathbb{R}^n$ [95% enough for our course]

In general $\Delta\vec{x} \to$ Tangent vectors at point x $T_x M$

$df \in$ linear functionals of Tangent vectors $T_x^* M$



c) Differential 1-form : $\omega(x)$ at each point X it is a linear functional on $\vec{\Delta x}$

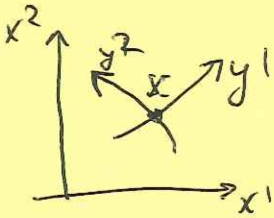
- ~~arbitrary linear functional (any element of $T^*_x M$)~~

$\omega: M \to T^*M$ You defined: $\underbrace{\omega: M \to}\, T^*M$ [Joke about French students]

$= \omega(x) = \sum_{i=1}^{n} \omega_i(x)\, dx^i$

Later **Important** values of $\omega_i(x)$ **depend** on coordinates chosen

Erase everything



$$\underbrace{\omega(X) = \sum_{i=1}^{n} \omega_i^{"x"}(X)\, dx^i}_{\text{at point } X \text{ in coordinates } \{x^1, x^2\}} = \left(\text{think about } x^i \text{ as functions } x^i(\bar{y})\right)$$

$$= \sum_{i=1}^{n} \omega_i(x)\left(\frac{\partial x^i}{\partial y^j}\, dy^j\right) = \underbrace{\omega_i(X)\frac{\partial x^i}{\partial y^j}}_{\omega_j^{"y"}(X)}\, dy^j =$$

$$= \underbrace{\omega^{"y"}_i\, dx^i}_{\text{at the same point } X \text{ in coordinates } \{y^1, y^2\}.}$$

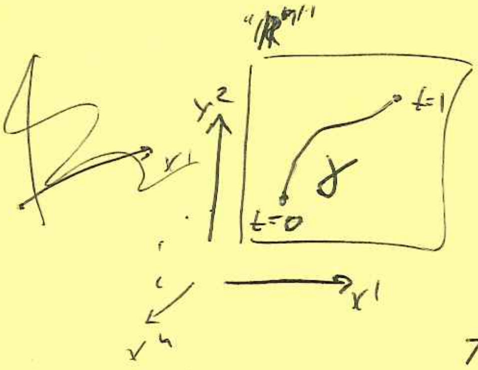
You can think about ω_i as a column, but its components depend on the chosen coordinate frame, exactly like components of vector depend on a chosen basis.

Sooner "a" $\equiv$ "c"

"b" is called exact differential

$= df$

Integration



$x^i(t)$

$$\int_\gamma \omega \equiv \int_0^1 \omega_i(x) \frac{\partial x^i}{\partial t} dt$$

This definition does not depend on a choise of coordinates

$$\int_0^1 \omega_i^{"x"} \frac{\partial x^i}{\partial t} dt = \int_0^1 \omega_j^{"y"} \underbrace{\frac{\partial y^j}{\partial x^i} \frac{\partial x^i}{\partial t}}_{\frac{\partial y^j}{\partial t}} dt = \int_0^1 \omega_i^{"y"} \frac{\partial y^i}{\partial t} dt$$

~~Important property:~~

Property of exact differential:

$$df = \frac{\partial f}{\partial x^i} dx^i \qquad \int_\gamma df = f(\vec{x}(t=1)) - f(\vec{x}(t=0))$$

$$\| $$

$$\int_0^1 \cancel{f} \underbrace{\frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial t}}_{\frac{df}{dt}} dt = \int_0^1 \frac{df}{dt} dt = f(t=1) - f(t=0)$$

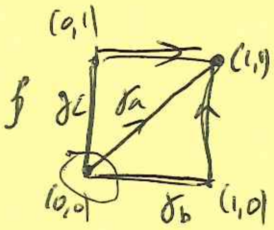
Consequence: – integral of exact differential does not depend on a path but only on the end points.

Examples:

$$\cancel{\alpha} = dx + dy = d(x+y)$$

$$\cancel{\beta} = y\,dx + dy \neq df$$

$$\frac{-y\,dx + x\,dy}{y^2} = d\left(-x/y\right)$$



$$\int_{\gamma_a} d(x+y) = x+y \Big|_{x=0,\ y=0}^{x=1,\ y=1} = 2 = \int_{\gamma_b} d(x+y)$$

$$\oint$$

γ_a: $x = t$, $y = t$

$$\int_{\gamma_a} d(-x/y) = \int_0^1 \left(\frac{x}{y^2}\frac{\partial y}{\partial t} - \frac{1}{y}\frac{\partial x}{\partial t}\right) dt = \int_0^1 \left(\frac{1}{t} - \frac{1}{t}\right) dt = 0$$

$$\int_{\gamma_b} d(-x/y) = \cancel{\int_{(0,0)}^{(1,0)}} \quad \text{not defined} = \infty$$

$$\int_{\gamma_c} d(-x/y) = \underbrace{\int_{(0,0)}^{(0,1)} d(-x/y)}_{x=0;\ y=t} + \int_{(0,1)}^{(1,1)} d(-x/y) =$$

$$= \int_0^1 0\,dt + \left(-x/y\right)\Big|_{x=0,\ y=1}^{x=1,\ y=1} = -1$$

$$\gamma_a = \int_0^1 (t+1)\,dt \qquad \int_{\gamma_a} y\,dx + dy = \int_0^1 (t+1)\,dt = \frac{1}{2} + 1 = 3/2$$

$$\int_{\gamma_b} y\,dx + dy = \underbrace{\int_{(0,0)}^{(1,0)} y\,dx + dy}_{x=0,\ y=t} + \underbrace{\int_{(1,0)}^{(t,1)} y\,\overset{dt}{dx} + \overset{0}{dy}}_{x=t,\ y=1} =$$

$$= \int 0\,dt + \int_0^1 dt = 1$$