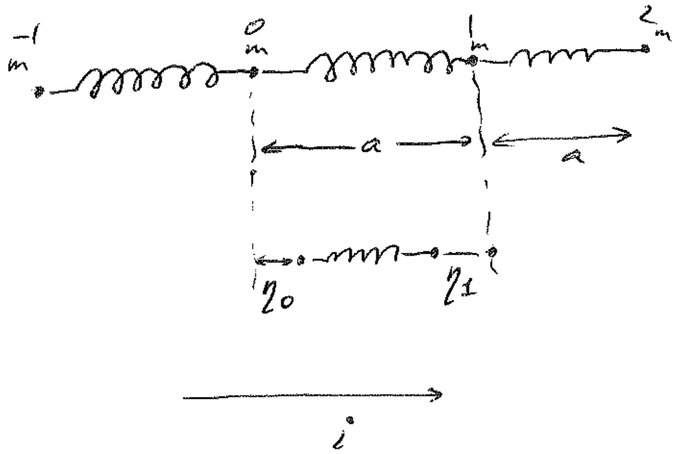
Lecture continuous systems

① Continuous system from discrete system:



$$\mathcal{L} = K - V$$

$$K = \frac{m}{2}\sum_i \dot{\eta}_i^2$$

$$V = \frac{1}{2}\sum_i k(\eta_{i+1}-\eta_i)^2$$

↑ force constant of a spring

~~Y = ka Youngs modulus~~

Take the continuous limit

$$K: \quad \frac{m}{2}\sum_i \dot{\eta}_i^2 = \frac{\overset{a}{\|}}{\Delta x}\cdot\frac{(m/a)}{2}\sum_i \dot{\eta}^2(x_i,t) \overset{a\to 0}{=} \frac{(m/a)}{2}\int dx\, \dot{\eta}^2$$

$x_i = ai$

$x = a\cdot i$

$\eta_i \to \eta(x)$

$\Delta x = a$

$a \to 0$

$m \to 0$

$m/a = \mu$ fixed

↑ mass per unit length

$$V: \quad \frac{1}{2}\sum_i k(\eta_{i+1}-\eta_i)^2 = \sum_i \Delta x \; \frac{ka}{2}\left(\frac{\eta_{i+1}-\eta_i}{a}\right)^2 \overset{a\to 0}{=} \int dx\, \frac{ka}{2}\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$\frac{\eta_{i+1}-\eta_i}{a} = \frac{\eta(x+a)-\eta(x)}{a} \;\overset{a\to 0}{=\!=}\; \frac{\partial\eta}{\partial x}$$

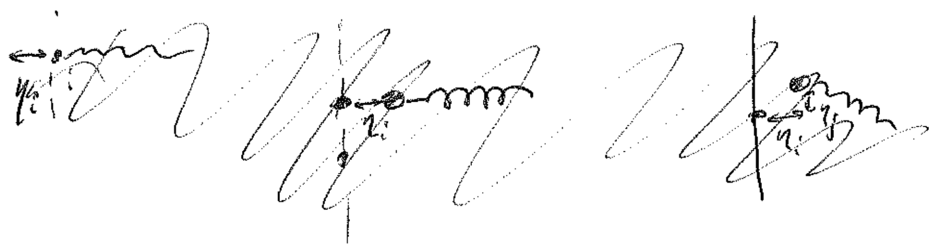
~~ka~~ = $a\to 0$, $k\to\infty$, $k\cdot a = Y$ - fixed (Youngs modulus)

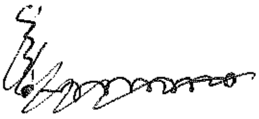
$$\mathcal{L} = \frac{1}{2}\int\left(\mu\dot{\eta}^2 - Y(\eta')^2\right)dx$$

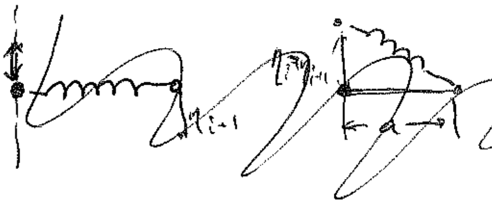
Important point: x here plays the role of coordinate labeling.

discrete system:	cont. system
q_i	$\eta(x)$ $q \leftrightarrow \eta$, $i \to x$

② Consider also transverse movements







$$\sqrt{(\eta_{i+1}-\eta_i)^2 + a^2} = a + \frac{1}{2a}(\eta_{i+1}-\eta_i)^2$$

$$\frac{1}{2} k \frac{(\eta_{i+1}-\eta_i)^4}{4a^2}$$

$$L = \iint \mathcal{L} \, dx dt$$

$\mathcal{L}$ – Lagrangian density ; $\mathcal{L} = \frac{1}{2}\left(\mu\left(\frac{\partial \eta}{\partial t}\right)^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2 \right)$

x and t play a similar role.

$$S = \iint \mathcal{L} \, dx \, dt$$

EOM from minimisation of S:

$$\delta S = \iint \delta(\mathcal{L}(\eta, \eta', \dot{\eta})) \, dx \, dt =$$

$$= \iint \left(\frac{\partial \mathcal{L}}{\partial \eta} \delta\eta + \frac{\partial \mathcal{L}}{\partial \eta'} \delta\eta' + \frac{\partial \mathcal{L}}{\partial \dot{\eta}} \delta\dot{\eta} \right) dx dt =$$

$$= \iint \left(\frac{\partial \mathcal{L}}{\partial \eta} \delta\eta + \left(-\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \eta'}\right)\right)\delta\eta + \left(-\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) \cdot \delta\eta \right) dx dt.$$

EOM: $$\boxed{\frac{\partial}{\partial t} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\eta}} + \frac{\partial}{\partial x} \cdot \frac{\partial \mathcal{L}}{\partial \eta'} - \frac{\partial \mathcal{L}}{\partial \eta} = 0}$$

(3) In general, let us have one time coordinate. Denote it $x^0 = ct$ and D space coordinates. Denote them x^i $\Big\} x^\mu$

(11/3)

So far it is not a relativistic theory, it is just a parameterisation

$\mathcal{L}(\eta, \partial_\mu \eta)$

η - analog of q

x^0 - analog of time

x^i - analog of i in q_i

$$S = \iiint_{\text{boundary conditions}} \mathcal{L}(\eta, \partial_\mu \eta)\, dx^0 d^D x$$

$$\delta S = 0 \Rightarrow \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta)} \right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0}$$

↑ summation over μ

Now compute explicitly for elastic rod:

$$\frac{\partial L}{\partial \eta} = 0; \quad \frac{\partial \mathcal{L}}{\partial \dot{\eta}} = \mu \dot{\eta}; \quad \frac{\partial \mathcal{L}}{\partial \eta'} = Y \eta'$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \eta'}\right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0$$

$$\mu \frac{\partial^2 \eta}{\partial t^2} - Y \frac{\partial^2 \eta}{\partial \eta^2} = 0$$ [Klein-Gordon equation]

Describes propagation of waves with speed $v = \sqrt{\frac{Y}{\mu}}$

- Mechanical background is not essential
- Describes evolution of field
- Describes wave excitations in continuous media
- Different kind of wave

④ In Lagrangian mechanics, there was an important quantity - E that is conserved.

Possible derivation:

$$\frac{d}{dt}\mathcal{L}(q,\dot q) = \underbrace{\frac{\partial \mathcal{L}}{\partial q}}_{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right)}\dot q + \frac{\partial \mathcal{L}}{\partial \dot q}\ddot q = \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right)\right]\dot q + \left(\frac{\partial \mathcal{L}}{\partial \dot q}\right)\cdot\frac{d}{dt}\dot q =$$

$$= \frac{d}{dt}\left(\dot q\,\frac{\partial \mathcal{L}}{\partial \dot q}\right)$$

$$\frac{d}{dt}\left(\dot q\frac{\partial \mathcal{L}}{\partial \dot q} - \mathcal{L}\right) = 0 \implies E = \dot q\frac{\partial \mathcal{L}}{\partial \dot q} - \mathcal{L} \text{ is conserved.}$$

Continous system version:

$$\partial_\nu(\delta_\mu^\nu \mathcal{L}) = \partial_\mu \mathcal{L}(\eta, \partial_\nu \eta) = \underbrace{\frac{\partial \mathcal{L}}{\partial \eta}}_{\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \eta}\right)}\cdot\partial_\mu \eta + \frac{\partial \mathcal{L}}{\partial \partial_\nu \eta}\partial_\mu\partial_\nu \eta =$$

$$= \left[\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \eta}\right)\right]\cdot\partial_\mu \eta + \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \eta}\right)\cdot\partial_\nu(\partial_\mu \eta) =$$

$$= \partial_\nu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \eta)}\partial_\mu \eta\right]$$

$$T_\mu{}^\nu = (\partial_\mu \eta)\cdot\frac{\partial \mathcal{L}}{\partial(\partial_\nu \eta)} - \delta_\mu{}^\nu \mathcal{L}$$

$$\boxed{\partial_\nu T_\mu{}^\nu = 0}$$

Analog $\partial_\nu j^\nu = 0 \Leftrightarrow$ continuity equation

$$j^\nu = (\rho, \vec j)$$

$$\frac{\partial \rho}{\partial x^0} + \operatorname{div}\vec j = 0 \qquad \frac{d}{dx^0}\int \rho\, d^D x = \int\left(\frac{\partial \rho}{\partial x^0}\right) d^D x =$$

$$= \int_{\Sigma} -\operatorname{div}\vec j\, d^D x = -\int_{\partial\Sigma} \vec j\, d\vec S = 0$$

(5) $\int T_\mu{}^0 \, d^D x$ – is a conserved quantity.

$$\int T_0{}^0 \, d^D x = \int \underbrace{\left(\dot\eta \frac{\partial \mathcal{L}}{\partial \dot\eta} - \mathcal{L} \right)}_{\text{energy density.}} d^D x \quad \leftrightarrow \quad \sum_i \dot q_i \frac{\partial \mathcal{L}}{\partial \dot q_i} - \mathcal{L} \quad \left[\begin{matrix} \text{Total} \\ \text{Energy} \end{matrix} \right]$$

$$\int T_i{}^0 \, d^D x = \int (\partial_i \eta) \frac{\partial \mathcal{L}}{\partial \dot\eta}$$

$$\int \partial_x \eta \, \frac{\partial \mathcal{L}}{\partial \dot\eta} \, dx \quad \leftrightarrow \quad [illegible] \sum_i (q_{i+1} - q_i) \underbrace{\frac{\partial \mathcal{L}}{\partial \dot q_i}}_{p_i}$$