Notes on canonical transformations

First point of view on canonical transformations. By definition, a change of variables $Q = Q(q,p)$, $P = P(p,q)$ is called canonical transformation if it preserves the symplectic structure. The most invariant way to look on the canonical transformation is to introduce a unique letter for the collection of both q's and p's: $x^\alpha = (q,p)$, and respectively $X^\alpha = (Q,P)$. In these notations the symplectic structure is a $2n \times 2n$ antisymmetric matrix $\omega_{\alpha\beta}$. This matrix is a vector-type object, more precisely a rank 2 covariant tensor, this means that the explicit values of its components depend on the coordinate system we use. The rule of transformation between different coordinate systems reads[1]

$$\omega^{"x"}_{\alpha\beta} = \frac{\partial X^{\alpha'}}{\partial x^\alpha}\frac{\partial X^{\beta'}}{\partial x^\beta}\omega^{"X"}_{\alpha'\beta'}\,, \tag{1}$$

where superscripts $^{"x"}$ and $^{"X"}$ precise in what coordinate system the components of ω are considered. The matrix $J^{\alpha'}{}_\alpha = \frac{\partial X^{\alpha'}}{\partial x^\alpha}$ is what is usually called the Jacobi matrix.

The invariance of symplectic structure simply means

$$\omega^{"x"}_{\alpha\beta} = \omega^{"X"}_{\alpha\beta}\,. \tag{2}$$

[Transformation (**??**) is very common. For instance, in special relativity, let x be the time-space coordinate in one inertial frame and X be the time-space coordinates in another inertial frame. Two frames are related by the Lorentz transformation: $X^\mu = L^\mu{}_\nu x^\nu$. The Jacobi matrix of this transformation is, obviously, $J = L$. We know that vectorial objects are "rotated" by L when we go from one frame to another. In particular, the electromagnetic tensor $F_{\mu\nu}$ is a covariant rank 2 tensor (btw, it is antisymmetric, as ω), and it transforms precisely as (**??**):

$$F_{\mu\nu} = L^{\mu'}{}_\mu L^{\nu'}{}_\nu F'_{\mu'\nu'}\,, \tag{3}$$

F' stands for the F in the X-system. In index-free notation the last equation is

$$F = L^{\mathrm{T}} \cdot F' \cdot L\,, \qquad \text{or} \qquad F' = (L^{\mathrm{T}})^{-1} \cdot F \cdot L^{-1}\,. \tag{4}$$

]

The primary interest in the canonical transformation is that equations of motion look exactly the same in old and new variables. Indeed, equations of motion in the x-language read

$$\frac{dx^\alpha}{dt} = \omega^{\alpha\beta}\frac{\partial\mathcal{H}}{\partial x^\beta}\,, \tag{5}$$

where $\omega^{\alpha\beta} \equiv (\omega^{-1})^{\alpha\beta}$ so that $\omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha{}_\gamma$.

Provided that the transformation does not depend on time we can directly perform the change of variables and write this equation in the new coordinates:

$$(J^{-1})^\alpha{}_{\alpha'}\frac{dX^{\alpha'}}{dt} = \omega^{\alpha\beta}J^\gamma{}_\beta\frac{\partial\mathcal{H}}{\partial X^\gamma} \tag{6}$$

[1]For those who studied exterior forms: ω is actually a 2-form: $\omega = \sum_{\alpha<\beta}\omega_{\alpha\beta}\,dx^\alpha \wedge dx^\beta$. Transformation (**??**) is the pullback $\pi^*(\omega)$ induced by the map $\pi : x \mapsto X$.

or

$$\frac{dX^\alpha}{dt} = J^\alpha{}_{\alpha'} J^\gamma{}_\beta \omega^{\alpha'\beta} \frac{\partial \mathcal{H}}{\partial X^\gamma} \quad \to \quad (J^\alpha{}_{\alpha'} J^\gamma{}_\beta \omega^{\alpha'\beta} = \omega^{\alpha\gamma}) \to \boxed{\tfrac{dX^\alpha}{dt} = \omega^{\alpha\beta} \tfrac{\partial \mathcal{H}}{\partial X^\beta}} \tag{7}$$

Equation in the box has exactly the same form as (**??**). Later we generalise the statement of invariance of equations to the time-dependent transformations.

Invariance of symplectic structure automatically implies invariance of the phase volume under canonical transformations, or, equivalently, of the measure of integration[2] $\prod_{i=1}^n dq^i\, dp_i = \prod_{i=1}^n dQ^i\, dP_i$. Indeed, $|\det J|$ is the Jacobian of transformation between two measures. But since $J \cdot \omega^{-1} \cdot J^{\mathrm{T}} = \omega^{-1}$, see (**??**), we conclude that $|\det J| = 1$.

Recall that evolution of $x = (q, p)$ under equations of motion can be considered, at a given moment of time, as a canonical transformation between coordinates - initial conditions x_0 and coordinates at a given moment of time. I.e. if evolution is $x(x_0, t)$ then for some $t = \tau$ one relates X and x_0 as $X = x(x_0, \tau)$. We know that, in particular, equations of motion preserve the phase volume of the system (Liouville's theorem).

There is an instructive way to derive transformation rule (**??**). The Poisson brackets are defined as

$$\{f, g\}_x \equiv \omega^{\alpha'\beta'}_{\text{"}x\text{"}} \frac{\partial f}{\partial x^{\alpha'}} \frac{\partial g}{\partial x^{\beta'}} \,. \tag{8}$$

Note that the definition is sensible to the coordinate system because we take derivatives with respect x's in very particular coordinate system, and ω itself depends on the coordinate system. That is why we put subscript $_x$ near the Poisson bracket. Now, if we request that the value of the Poisson bracket does not depend on the coordinate system, this gives us the rule how ω transforms (more precisely, its inverse $\omega^{\alpha\beta}$):

$$\{f, g\}_X = \omega^{\alpha\beta}_{\text{"}X\text{"}} \frac{\partial f}{\partial X^\alpha} \frac{\partial g}{\partial X^\beta} = \omega^{\alpha\beta}_{\text{"}X\text{"}} \left(\frac{\partial x^{\alpha'}}{\partial X^\alpha} \frac{\partial f}{\partial x^{\alpha'}} \right) \left(\frac{\partial x^{\beta'}}{\partial X^\beta} \frac{\partial g}{\partial x^{\beta'}} \right) \tag{9}$$

By requiring that $\{f, g\}_x = \{f, g\}_X$, we get from (**??**) and (**??**) that

$$\omega^{\alpha'\beta'}_{\text{"}x\text{"}} = \omega^{\alpha\beta}_{\text{"}X\text{"}} \frac{\partial x^{\alpha'}}{\partial X^\alpha} \frac{\partial x^{\beta'}}{\partial X^\beta}, \tag{10}$$

which is equivalent to (**??**). For canonical transformation, $\omega^{\alpha\beta}_{\text{"}x\text{"}} = \omega^{\alpha\beta}_{\text{"}X\text{"}}$, so we can write down the demand for canonical transformation as $\omega^{\alpha'\beta'} = \omega^{\alpha\beta} \frac{\partial x^{\alpha'}}{\partial X^\alpha} \frac{\partial x^{\beta'}}{\partial X^\beta}$, or equivalently as

$$\omega^{\alpha\beta} = \omega^{\alpha'\beta'} \frac{\partial X^\alpha}{\partial x^{\alpha'}} \frac{\partial X^\beta}{\partial x^{\beta'}} = \begin{pmatrix} \text{by definition of} \\ \text{Poisson bracket} \end{pmatrix} = \{X^\alpha, X^\beta\} \tag{11}$$

Note that Poisson bracket is computed in the x-coordinates, X^α is considered as function of x's here.

Condition $\{X^\alpha, X^\beta\} = \omega^{\alpha\beta}$ is the necessary and sufficient condition for the transformation to be canonical. It can be explicitly used in the exercises. When we write down explicitly in terms of P and Q, this condition becomes

$$\{Q^i, P_j\} = \delta^i{}_j \,, \qquad \{Q^i, Q^j\} = 0 \qquad \{P_i, P_j\} = 0 \,. \tag{12}$$

[2]In the point of view when x and X are coordinates of different spaces and the canonical transformation is the map $\pi : x \mapsto X$, and the volume is preserved in the following sense: in system x the volume of a domain D is $V_x = \int_D d\mu_x$ with $d\mu_x = \prod_{i=1}^n dq^i\, dp_i$. The domain D mapped to the domain $\pi(D)$ in the system X. Its volume should be computed as $V_X = \int_{\pi(D)} d\mu_X$ with $d\mu_X = \prod_{i=1}^n dQ^i\, dP_i$. The statement is that $V_x = V_X$.

Once more, Poisson bracket is computed in the (q,p)-frame, e.g. $\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}$, written above equalities is not a tautology.

Second point of view on canonical transformations. As above, we do not consider time-dependent canonical transformations so far. We can think about P, Q as a different way to parameterise **the same** 2n-dimensional phase space, which is originally parameterised by p, q. In this 2n-dimensional phase space we can introduce differential 1-forms, for instance

$$\sum_{i=1}^{n} p_i \, dq^i - P_i \, dQ^i \tag{13}$$

In the following I will write down only $n = 1$ case, for simplicity. So the form above is $p\,dq - P\,dQ$.

If we think about P and Q as functions of p and q, then $dQ = \frac{\partial Q}{\partial p}_{|q} dp + \frac{\partial Q}{\partial q}_{|p} dq$. But, just to note, we could think about Q as a function of any 2 variables. For instance, we can consider p and P (in non-singular cases) as two variables we want to use. Then $dQ = \frac{\partial Q}{\partial p}_{|P} dp + \frac{\partial Q}{\partial P}_{|p} dP$. Note that $\frac{\partial Q}{\partial p}_{|q} \neq \frac{\partial Q}{\partial p}_{|P}$.

Partial derivative measures the change of a function in certain direction. $_{|q}$ *means that along the chosen direction q is fixed.* $_{|P}$ *means that along the chosen direction P is fixed. These are different directions, so the change of function would be different!*

Canonical transformation is the transformation $P = P(p,q)$, $Q = Q(p,q)$ such that $p\,dq - P\,dQ$ is the exact differential:

$$dS_1 = p\,dq - P\,dQ \tag{14}$$

S_1 is called the generating function of the canonical transformation. Given expression (**??**), it is the easiest to consider S_1 as a function of q, Q. Then it is immediate:

$$\frac{\partial S_1}{\partial q}_{|Q} = p\,, \qquad \frac{\partial S_1}{\partial Q}_{|q} = -P\,. \tag{15}$$

Note that if $p\,dq - P\,dQ$ is exact then e.g. $-q\,dp - P\,dQ$ is exact. Indeed, $d(S_1 - q\,p) = p\,dq - P\,dQ - d(q\,p) = p\,dq - P\,dQ - p\,dq - q\,dp = -q\,dp - P\,dQ$. The function $S_2 = S_1 - q\,p$ is also called the generating function (of the second kind). S_2 is nothing but a Legendre transform, with proper sign chosen, of S_1. Using the same logic, we introduce $S_3 = S_1 + Q\,P$ and $S_4 = S_1 + Q\,P - q\,p$. In total we have

$$\begin{aligned} dS_1 &= +p\,dq - P\,dQ \qquad (16)\\ dS_2 &= -q\,dp - P\,dQ \qquad (17)\\ dS_3 &= +p\,dq + Q\,dP \qquad (18)\\ dS_4 &= -q\,dp + Q\,dP\,. \qquad (19) \end{aligned}$$

Equivalence. The discussed two points of view on the canonical transformations are equivalent. Indeed, you can show the following statement[3]

$$\{Q,P\}=1 \quad \text{is equivalent to} \quad -\frac{\partial P}{\partial q}_{|Q}=\frac{\partial p}{\partial Q}_{|q} \tag{20}$$

This equivalence is proven in the last question of the tutorial 4.

The condition on the left is the condition of invariance of the symplectic structure.

The condition on the right is the condition of exactness of the differential $p\,dq - P\,dQ$.

Dependence on time. If a canonical transformation has explicit dependence on time, then the equations of motion are not covariant in a way how it was for time-independent case. In detail, if one has $X = X(x,t)$, then from equation $\frac{dX^\alpha}{dt}=\omega^{\alpha\beta}\frac{\partial\mathcal{H}'}{\partial X^\beta}$ we will get

$$J^\alpha{}_{\alpha'}\frac{dx^{\alpha'}}{dt}+\frac{\partial X^\alpha}{\partial t}=J^\alpha{}_{\alpha'}\omega^{\alpha'\beta}\frac{\partial\mathcal{H}'}{\partial x^\beta}$$

or equivalently

$$\frac{dx^\alpha}{dt}=\omega^{\alpha\beta}\frac{\partial\mathcal{H}'}{\partial x^\beta}-\frac{\partial x^\alpha}{\partial X^\gamma}\frac{\partial X^\gamma}{\partial t}\,. \tag{21}$$

The last term spoils covariance! However, it appears that $\frac{\partial x^\alpha}{\partial X^\gamma}\frac{\partial X^\gamma}{\partial t}$ defines a symplectic (Hamiltonian) vector flow (proof is below). This means that there exists such K that $\frac{\partial x^\alpha}{\partial X^\gamma}\frac{\partial X^\gamma}{\partial t}=\omega^{\alpha\beta}\frac{\partial K}{\partial x^\beta}$. Hence, for $\mathcal{H}'=\mathcal{H}+K$ we will get the usual equations of motion

$$\frac{dx^\alpha}{dt}=\omega^{\alpha\beta}\frac{\partial\mathcal{H}}{\partial x^\beta}\,. \tag{22}$$

Conclusion: time-dependent canonical transformation also changes Hamiltonian of the system ($\mathcal{H}\to\mathcal{H}'$). On the language of the generating function one can show that this statement transforms to the statement that

$$dS_1 = p\,dq - P\,dQ - (\mathcal{H}-\mathcal{H}')dt\,. \tag{23}$$

I.e. $p\,dq - P\,dQ$ is no longer exact in the extended phase space of p, q and t, but the combination in the formula above is exact. The last equation is the practical way to find $\mathcal{H}'$.

The minimum requirement for you is to know (**??**)*, its meaning, and how to use it (e.g. connection to Hamilton-Jacobi equation). Details of derivations are not compulsory.*

Let us prove now that $\frac{\partial x^\alpha}{\partial X^\gamma}\frac{\partial X^\gamma}{\partial t}$ is a symplectic vector flow. Reminder: vector field v^α defines the symplectic flow if $\omega_{\alpha\gamma}v^\gamma$ is exact. The last we check by computing $\partial_\alpha\omega_{\beta\gamma}v^\gamma-\partial_\beta\omega_{\alpha\gamma}v^\gamma$ and checking whether it is 0. Let us apply this program to $v^\alpha=\frac{\partial x^\alpha}{\partial X^\gamma}\frac{\partial X^\gamma}{\partial t}$. We use $\omega_{\alpha\beta}\frac{\partial x^\beta}{\partial X^\gamma}=\frac{\partial X^\beta}{\partial x^\alpha}\omega_{\beta\gamma}$, which is a consequence of (**??**), to get $\omega_{\alpha s}v^s=\frac{\partial X^\beta}{\partial x^\alpha}\omega_{\beta\gamma}\frac{\partial X^\gamma}{\partial t}$. Then

$$\partial_\alpha\omega_{\beta s}v^s-\partial_\beta\omega_{\alpha s}v^s=\frac{\partial}{\partial x^\alpha}\left(\frac{\partial X^\delta}{\partial x^\beta}\omega_{\delta\gamma}\frac{\partial X^\gamma}{\partial t}\right)-\frac{\partial}{\partial x^\beta}\left(\frac{\partial X^\delta}{\partial x^\alpha}\omega_{\delta\gamma}\frac{\partial X^\gamma}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\frac{\partial X^\delta}{\partial x^\beta}\omega_{\delta\gamma}\frac{\partial X^\gamma}{\partial x^\alpha}\right)=\frac{\partial\omega_{\alpha\beta}}{\partial t}=0\,, \tag{24}$$

where we used that ω is anti-symmetric and time-independent.

[3] For those who are familiar with exterior forms. In the canonical coordinates, $\omega = dp\wedge dq$. So one can write $\omega = d\alpha$, where $\alpha = p\,dq$, i.e. ω is d-exact.

Note aside: $d\omega=0$ is true in any basis, not only canonical. Symplectic structure, strictly speaking, is defined as a non-degenerate ($\det\limits_{\alpha,\beta}\omega_{\alpha\beta}\neq 0$) d-closed ($d\omega=0$) differential 2-form. Condition $d\omega=0$ is equivalent to the Jacobi identity for the Poisson brackets defined by $\omega^{\alpha\beta}$. Exactness of ω ($\omega=d\alpha$) is not always true, depends on the topology of the manifold. It is true for trivial topologies, we discuss only such topologies.

Canonical transformation preserves ω which means $\omega-\pi^*(\omega)=0$. But the last equality can be also written as $d(p\,dq-P\,dQ)=0$. This means that $p\,dq-P\,dQ$ is d-exact, as requested. The opposite is obvious.