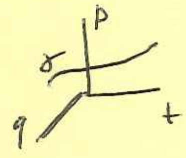
$$S' = \int p\,dq - H dt \quad ; \quad \delta S = 0 \;\rightarrow\; \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \; ; \; \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

* ? In which space do we integrate?

t ?	Answer: $\int_{t_1}^{t_2} \left(p(t) \frac{dq(t)}{dt} - \mathcal{H}(p(t), q(t), t) \right) dt$
q, t ?	$\delta S = 0$ means : For what functions $q(t), p(t)$ of time t S' has extremum?
p, q, t ? ✓	



$$\int_{\gamma} (p\,dq - \mathcal{H} dt)$$

$\delta S = 0$ means: For what <u>path</u> in p, q, t space S' has extremum?

* ? what should I fix at the end points

$t_i \; ; \; t_f$?	Answer: we certainly fix time
$t_i, q_i \; ; \; t_f, q_f$? ✓	and we need only 2 other parameters to define motion.
$t_i, q_i, p_i \; ; \; t_f, q_f, p_f$?	Here, however, our only choice is <u>q</u>

$$\delta S = \int \delta p\, dq + p\, d\delta q - \dots =$$

$$= \int \delta p\, dq + \underbrace{p\,\delta q \Big|_{init}^{final}}_{=0} - \int \delta q\, dp + \dots$$

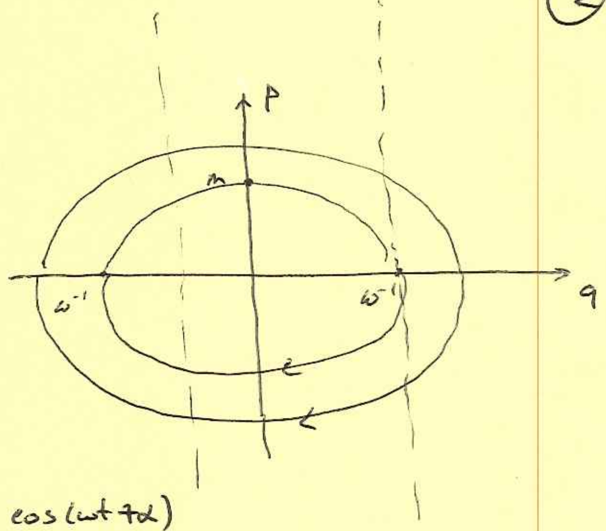
* ? Does value of integration depend on the path

Of course, it does!

<u>However</u>, we can think about $S(q_f, t_f; q_i, t_i)$ in the following sense. For a given $q_f, t_f; q_i, t_i$ ~~find~~ solve EOM $\equiv$ find $\gamma_{extremum}$

$$S(q_f, t_f; q_i, t_i) = \int_{\gamma_{extremum\ path}} p\,dq - \mathcal{H} dt$$

Example: Harmonic oscillator.



$$\mathcal{H} = \cancel{\tfrac{1}{2}\,\tfrac{p^2}{}} \; \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 =$$

$$= \frac{m}{2}\left(\left(P/m\right)^2 + (\omega q)^2\right)$$

$$p = \cos(\omega t)\, p_i - \sin(\omega t)\, m\omega q_i = mA\cos(\omega t + \alpha)$$

$$q = \sin(\omega t)\, \frac{p_i}{m\omega} + \cos(\omega t)\, q_i = \frac{1}{\omega} A \sin(\omega t + \alpha)$$

$$\underline{\underline{A^2}} = (P/m)^2 + (\omega q)^2 = \underline{\underline{\frac{2E}{m}}}$$

$t_i = 0;\ t_f = \tau$

$S(q_f, q_i, \tau)$

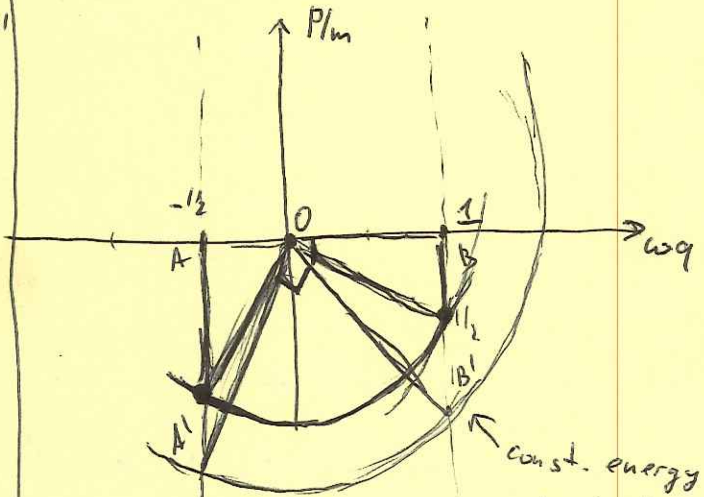
$q_i = \omega^{-1};\ q_f = -\tfrac{1}{2}\omega^{-1}$

$\tau = \frac{1}{4}\left(\frac{2\pi}{\omega}\right)$ ← time of one turn on the phase portrait.

$$S = \int p\,dq - \underbrace{\mathcal{H}dt}_{= E\,dt} = \frac{1}{2}\int d(pq) = \frac{p_f q_f - p_i q_i}{2} \quad (*)$$

$$p\,dq = \underbrace{\tfrac{1}{2}(p\,dq + q\,dp)}_{\frac{1}{2} d(pq)} + \underbrace{\tfrac{1}{2}(p\,dq - q\,dp)}_{\frac{1}{2} m A^2 dt \;=\; E\,dt}$$

$$\left.\begin{aligned} p_f &= \cos(\omega\tau)\, p_i - \sin(\omega\tau)\, m\omega q_i \\ q_f &= \sin(\omega\tau)\, p_i/m\omega + \cos(\omega\tau)\, q_i \end{aligned}\right\} \to \text{solve for } p_f;\ p_i$$



$\Delta OA'A = \Delta OB'B$

substitute $p_f; p_i$ to (*).

Answer: $\left\{\begin{aligned} p_i &= \frac{m\omega}{\sin(\omega\tau)}\left(q_f - \cos(\omega\tau)\, q_i\right) \\ p_f &= \frac{m\omega}{\sin(\omega\tau)}\left(\cos(\omega\tau)\, q_f - q_i\right) \end{aligned}\right.$

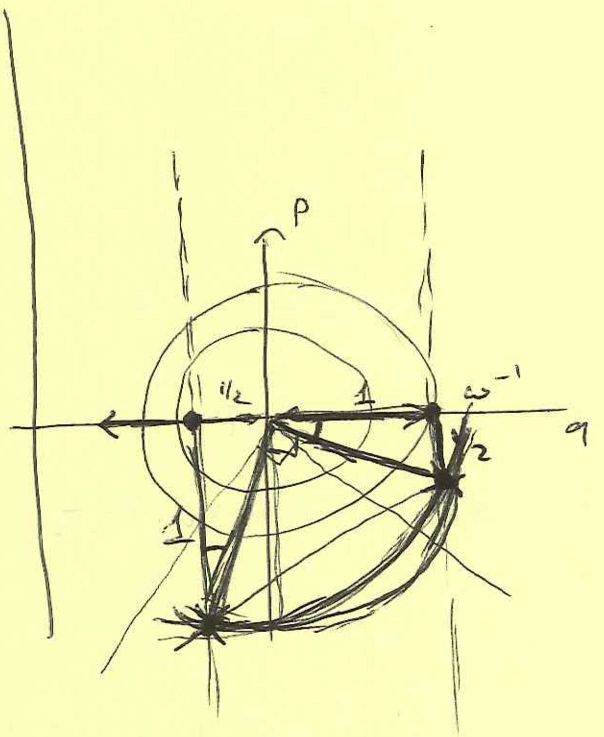
$$\boxed{S(q_f, q_i, \tau) = \frac{m\omega}{2\sin(\omega\tau)}\left[\cos(\omega\tau)\left[q_f^2 + q_i^2\right] - 2q_i q_f\right]}$$

$$p_i = \frac{m\omega}{\sin(\omega\tau)} \left(q_f - \cos(\omega\tau) q_i \right)$$

$$p_f = + \frac{m\omega}{\sin(\omega\tau)} \cancel{q_f} \left(\cos(\omega\tau) q_f - q_i \right)$$

$$\frac{1}{2}(p_f q_f - p_i q_i) =$$

$$= \frac{m\omega}{2\sin(\omega\tau)} \Big[\cos(\omega\tau) \left[q_f^2 + q_i^2 \right] - 2 q_i q_f \Big]$$



Consider action as a generator of canonical transformation

3

$\cancel{S(q,Q,t)} =$

$$S(q,Q,t) = \frac{m\omega}{2\sin(\omega t)} \Big[\cos(\omega t) \left[q^2 + Q^2 \right] - 2qQ \Big]$$

$$p = \frac{\partial S}{\partial q} = \frac{m\omega}{\sin(\omega t)} \left[\cos(\omega t) q - Q \right]$$

$$P = -\frac{\partial S}{\partial q} = \frac{m\omega}{\sin(\omega t)} \left[-\cos(\omega t) Q + q \right]$$

$\rightarrow$ solve for Q, P

$\cancel{Q = \cos(\omega t) q - \sin(\omega t) \frac{p}{m\omega}}$

$$\begin{cases} P = \cos(\omega t) p + \sin(\omega t) m\omega q \\ Q = -\sin(\omega t) \frac{p}{m\omega} + \cos(\omega t) q \end{cases}$$

Solution of equations of motion ~~with~~ at negative time.

$$\begin{cases} p = \cos(\omega t) P - \sin(\omega t) m\omega Q \\ q = -\sin(\omega t) \frac{Q}{m\omega} + \cos(\omega t) P \end{cases}$$

Action is a generating function of $S(q,Q,t)$ for a canonical transformation which sends q,p to the Q,P — initial conditions

$$H_{p,q} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

$$H'_{P,Q} = ?$$

$$p\,dq - P\,dQ - (H - H')\,dt = dS$$

$$\cancel{H - H' =}\quad H - H' = -\frac{\partial S}{\partial t}$$

$$H' = \cancel{H +}\frac{\partial S}{\partial t} + H = 0 \quad \cancel{\frac{m\omega^2}{2\sin(\omega t)}} \quad -\omega\cot(\omega t) S - \frac{m\omega^2}{2}(q^2+Q^2) =$$

$$\cancel{\frac{H}{S} = -\frac{m\omega^2}{2}\left[\cot^2(\omega t)(q^2+Q^2) - \frac{\cot(\omega t)}{\sin(\omega t)} 2qQ + (q^2+Q^2)\right]}$$

$\underline{\dot{P}} = 0$; $\dot{Q} = 0$ $\rightarrow$ $P, Q = \text{const}$: they are original coordinates ~~argu~~

__Formal derivation:__

We consider the case when H is not a function of time.

$$\cancel{S(q, Q, t) = \int_{q(t)=q_f}}$$

(*) $S(q_f, Q, \underbrace{t_f - t_i}_{t}) = \int_{q(0)=Q}^{q(t)=q_f} p\,dq - H\,dt$ ← integration over γ_{extremum}.

Consider $S_1(q, Q, t)$ such that $\frac{\partial S}{\partial t} + H = 0 \Leftrightarrow H' = 0$

$$dS_1 = p\,dq - P\,dQ - (H - \underbrace{H'}_{0})\,dt$$

$\int_{\cancel{q(0)=Q}}^{\cancel{q(t)=q_f}} dS_1$ ← depends only on final and initial points,
path can be any

So 1. Choose final and initial points as in (*)
2. Choose path γ_{extremum}.
Along this path $dQ = 0$ →

$$\rightarrow \int dS_1 = \int_{\gamma_{\text{extremum}}} p\,dq - H = S_1(q_f, q_i, t) \quad \text{by construction.}$$

$$\frac{\partial S}{\partial t} + \mathcal{H}(q, p) = 0$$

$$p = \frac{\partial S}{\partial q}$$

$$\frac{\partial S}{\partial t} + \mathcal{H}\left(q, \frac{\partial S}{\partial q}\right) = 0$$ ← Hamilton–Jacobi equation.

One way to find S is to solve HJ. If $\mathcal{H} = \frac{\vec{p}^2}{2m} + V(\vec{x})$

$$\boxed{\frac{\partial S}{\partial t} + \frac{1}{2m}(\vec{\nabla} S)^2 + V(x) = 0}$$

2. Reminiscense with ~~wave equation~~ wave-front equation

Wave:

$$\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right)\Psi = 0 \quad ;$$

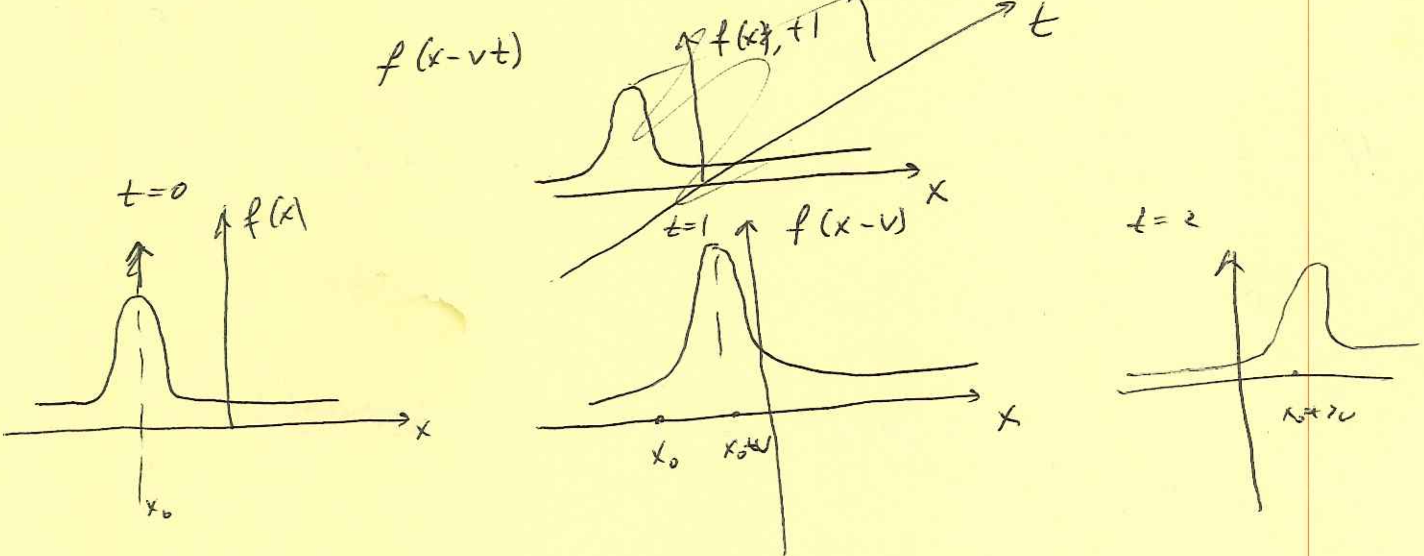
$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\Psi = 0$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)(x - vt) = 0$$

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)(x + vt) = 0$$

$$A = f(x - vt, x + vt)$$

$f(x-vt)$



$$\Psi_\omega = C\, e^{i\omega(x-vt)} \qquad \Psi = \sum C_{\omega_+} e^{i\omega_+(x-vt)} + \sum D_\omega e^{i\omega\ldots}$$